DEGENERATE AFFINE FLAG VARIETIES

AND

QUIVER GRASSMANNIANS

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Contents

Introduction

Motivation

Quiver Grassmannians where first used by W. Crawley-Boevey and A. Schofield [**23, 68**]. They are linked to cluster algebras as introduced by S. Fomin and A. Zelevinsky [**32**]. The cluster variables admit a description based on the Euler characteristic of quiver Grassmannians as shown by P. Caldero and F. Chapoton [**14**]. By now there are may publications concerning quiver Grassmannians, their Euler characteristic and Poincaré polynomials. But most of the research restricts to quiver Grassmannians for Dynkin quivers. In this work we generalise some constructions by P. Caldero, S. Fedotov and M. Reineke [**15, 27, 62, 63**] to quiver Grassmannians for bounded quiver representations which are equivalent to modules over finite dimensional algebras.

On the one hand this thesis is based on the identification of the degenerate flag variety with a quiver Grassmannian for the equioriented quiver of type A as shown by G. Cerulli Irelli, E. Feigin and M. Reineke in [**20**]. On the other hand it is based on the construction using quiver Grassmannians for the loop quiver to give finite approximations of the degenerate affine Grassmannian which was introduced by E. Feigin, M. Finkelberg and M. Reineke in [**30**]. We generalise their constructions to describe finite approximations of linear degenerate affine flag varieties using quiver Grassmannians for the equioriented cycle. The linear degenerations of the affine flag variety are defined similar to the construction for the classical flag variety by G. Cerulli Irelli, X. Fang, E. Feigin, G. Fourier and M. Reineke [**19**].

In some special case quiver Grassmannians for the equioriented cycle were studied by N. Haupt [**42, 41**]. The variety of representations of the cycle was studied by G. Kempken [**48**]. J. Sauter studied the quiver flag variety for the equioriented cycle [**65**]. The Ringel-Hall algebra of the cyclic quiver was studied by A. Hubery [**44**]. Based on the work by G. Kempken and A. Hubery we derive statements about the geometry of quiver Grassmannians for the equioriented cycle and obtain a generalisation of a result by N. Haupt.

Summary of Main Results

The Grassmannian $\mathrm{Gr}_k(n)$ is the set of all k-dimensional subspaces of the vectorspace $\mathbb{C}^n$. On this variety the group of invertible matrices $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$ acts transitively. Let P be the stabiliser of any point in the Grassmannian. Then the Grassmannian is isomorphic to the quotient GL_n/P. This quotient construction has been generalised to a great extend for various types of groups for example algebraic groups or Kac-Moody groups and subgroups like (maximal) parabolic and Borel or (maximal) parahoric and Iwahori. The resulting quotients are called

Grassmannians or (full/partial) flag varieties depending on the type of the subgroup.

The main goal of this thesis is the study of the degenerate affine flag variety of type $\mathfrak{gl}_n$ via approximations by quiver Grassmannians for the equioriented cycle. Analogous to the classical setting the affine flag variety is defined as the quotient

$$\mathcal{F}l(\widehat{\mathfrak{gl}}_n) := \widehat{\mathrm{GL}}_n / \widehat{B}_n$$

where $\widehat{\mathrm{GL}}_n$ is the affine Kac-Moody group to the affine Kac-Moody algebra $\widehat{\mathfrak{gl}}_n$ and $\widehat{B}_n$ is the standard Iwahori subgroup of $\widehat{\mathrm{GL}}_n$ [**53**, Chapter XIII]. It is not necessary to know the precise definitions of these groups to understand the main part of this thesis. Just bear in mind the definition of the full flag variety as the quotient of the invertible matrices by the invertible upper triangular matrices. This variety admits an alternative description as the set of all chains of vector spaces where the dimension of the spaces increases by one for each inclusion.

The first step in the direction of approximations of the affine flag variety by quiver Grassmannians is an alternative description of the affine flag variety which is similar to the set of vector space chains in the classical setting. The affine flag variety is infinite dimensional such that we have to replace the finite dimensional vector space by some infinite dimensional objects. There are two approaches to this problem.

The more common construction is via lattice chains [**3, 36, 37**]. Let $\mathbb{C}((t))$ be the field of Laurent series and $\mathbb{C}[[t]] \subset \mathbb{C}((t))$ be the ring of formal power series, define $\Lambda := \mathbb{C}[[t]]^n$. A lattice $\mathcal{L} \subset \mathbb{C}((t))^n$ is a Λ-submodule such that there exists an integer $N \in \mathbb{Z}_{\geq 0}$ with $t^N\Lambda \subseteq \mathcal{L} \subseteq t^{-N}\Lambda$ and the quotient $t^{-N}\Lambda/\mathcal{L}$ is of finite rank over $\mathbb{C}$. Λ is called the standard lattice.

A (full periodic) lattice chain is a tuple of lattices $(\mathcal{L}_i)_{i=0}^{n-1}$ such that

$$\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{n-1} \subset t^{-1}\mathcal{L}_0$$

and each quotient $\mathcal{L}_{i+1}/\mathcal{L}_i$ is a $\mathbb{C}$-module of rank one. The affine flag variety of type $\mathfrak{gl}_n$ is in bijection with the set of full lattice chains in $\mathbb{C}((t))^n$. It is possible to define approximations and degenerations of the affine flag variety in this setting but we want to take a different path where the analogy to the classical setting is more visible.

The second construction is based on Sato Grassmannians [**30, 45**]. For $\ell \in \mathbb{Z}$ let V_ℓ be the vectorspace

$$V_\ell := \mathrm{span}(v_\ell, v_{\ell-1}, v_{\ell-2}, \dots)$$

which is a subspace of the infinite dimensional $\mathbb{C}$-vectorspace V with basis vectors v_i for $i \in \mathbb{Z}$. The Sato Grassmannian for $m \in \mathbb{Z}$ is defined as

$$\mathrm{SGr}_m := \big\{ U \subset V \ : \ \text{There exists a } \ell < m \text{ s.t. } V_\ell \subset U \text{ and } \dim U/V_\ell = m - \ell \ \big\}.$$

The vector spaces in the chains for the classical flag variety are elements of the Grassmannians $\mathrm{Gr}_k(n)$. Analogous we obtain a description of the affine flag variety as a set of cyclic chains where the vector spaces are elements of the Sato Grassmannians SGr_k.

PROPOSITION 1 ([**30, 45**]). The affine flag variety $\mathcal{F}l(\widehat{\mathfrak{gl}}_n)$ as subset in the product of Sato Grassmannians is described as

$$\mathcal{F}l(\widehat{\mathfrak{gl}}_n) \cong \left\{ (U_k)_{k=0}^{n-1} \in \prod_{k=0}^{n-1} \mathrm{SGr}_k \ : \ U_0 \subset U_1 \subset \ldots \subset U_{n-1} \subset s_n U_0 \right\}$$

where $s_n : V \to V$ maps v_i to v_{i+n}.

It is shown in [**29**] that the degenerate flag variety admits a description via vector space chains where the spaces are related by projections instead of inclusions. This construction is used to define linear degenerations of the flag variety and degenerate affine Grassmannians in [**19**] and [**30**]. Here we want to follow the same approach and degenerate the affine flag variety by replacing the inclusion relations for the chains of vector spaces with projections.

DEFINITION 1. The degenerate affine flag variety $\mathcal{F}l^a(\widehat{\mathfrak{gl}}_n)$ is defined as

$$\mathcal{F}l^a(\widehat{\mathfrak{gl}}_n) := \left\{ (U_k)_{k=0}^{n-1} \in \prod_{k=0}^{n-1} \mathrm{SGr}_k \ : \ pr_{i+1} U_i \subset U_{i+1}, \ pr_n U_{n-1} \subset s_n U_0 \right\}$$

where $\mathrm{pr}_i : V \to V$ is the projection of v_i to zero.

Later this degeneration is also refered to as the Feigin-degenerate affine flag variety since its definition is analogous to the description of the degenerate classical flag variety studied by E. Feigin in [**28, 29**]. Similar to the construction in [**30**] the linear degenerate affine flag varieties are defined by writing linear maps f_i instead of the projections pr_i. Most of the constructions below work in the setting of certain linear degenerations of the affine flag variety.

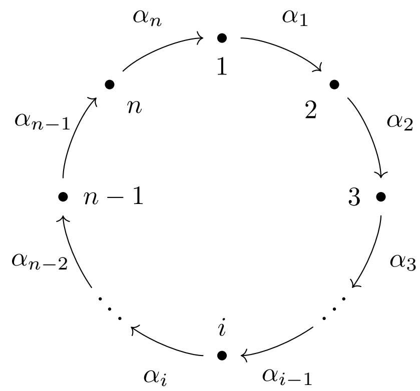
Let Q be a finite quiver with a finite set of vertices Q_0 and a finite set of arrows Q_1 between the vertices. A Q-representation R is a pair of tuples $R = (V, M)$ with a tuple of vector spaces over the vertices $V = (V_i)_{i \in Q_0}$ and a tuple of maps between the vector spaces along the arrows of the underlying quiver $M = (M_\alpha)_{\alpha \in Q_1}$.

A subrepresentation $S \subseteq R$ is described by a tuple of vector subspaces $U_i \subset V_i$ which is compatible with the maps between the vector spaces of the surrounding representation R, i.e. for all arrows $\alpha : i \to j$ of Q we have $M_\alpha(U_i) \subseteq U_j$. The entries of the dimension vector $\mathbf{dim}\, R \in \mathbb{Z}^{Q_0}$ of a quiver representation R are given by the dimension of the vector spaces V_i over the vertices of the quiver.

DEFINITION 2. The quiver Grassmannian $\mathrm{Gr}^Q_{\mathbf{e}}(M)$ is the set of all subrepresentations of the Q-representation M with dimension vector $\mathbf{e} \in \mathbb{Z}^{Q_0}$.

M. Reineke showed that every projective variety is a quiver Grassmannian [**64**]. Hence it makes sense to restrict the class of quiver Grassmannians which are considered. In this thesis we focus on quiver Grassmannians for the equioriented cycle:

$\Delta_n \ :=$



The set of vertices and arrows of Δ_n are in bijection with the set $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$.

These varieties yield finite dimensional approximations of the affine flag variety of type $\mathfrak{gl}_n$ and its linear degenerations. For a positive integer ω the finite approximation of the Feigin-degenerate affine flag variety is

$$\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big) = \Big\{(U_i)_{i=0}^n \in \mathcal{F}l^a\big(\widehat{\mathfrak{gl}}_n\big) \ : \ V_{-\omega n} \subset U_0 \subset V_{\omega n}\Big\}.$$

Theorem 1 (Theorem 6.4). Let $\omega \in \mathbb{N}$ be given, take the quiver representation

$$M_\omega := \Big(\big(V_i := \mathbb{C}^{2\omega n}\big)_{i\in\mathbb{Z}_n}, \big(M_{\alpha_i} := s_1 \circ \mathrm{pr}_{\omega n}\big)_{i\in\mathbb{Z}_n} \Big)$$

and the dimension vector $\mathbf{e}_\omega := (e_i := \omega n)_{i\in\mathbb{Z}_n}$. Then the finite dimensional approximation of the Feigin-degenerate affine flag variety is isomorphic to the quiver Grassmannian corresponding to M_ω and $\mathbf{e}_\omega$, i.e.

$$\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big) \cong \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}\big(M_\omega\big).$$

This construction allows us to obtain statements about the geometric properties of the approximations from the corresponding quiver Grassmannians.

Theorem 2 (Theorem 6.35). For $\omega \in \mathbb{N}$, the approximation $\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ of the Feigin-degenerate affine flag variety satisfies:

(1) It is a projective variety of dimension ωn^2.

(2) It admits a cellular decomposition.

(3) There is a bijection between the cells and affine Dellac configurations to the parameter ω.

The irreducible components of the finite dimensional approximation of the Feigin-degenerate affine flag variety satisfy:

(4) They are equidimensional.

(5) They have rational singularities and are normal, Cohen-Macaulay.

(6) There is a bijection between the irreducible components and grand Motzkin paths of length n.

Grand Motzkin paths of length n are lattice paths from $(0,0)$ to $(n,0)$ with steps $(1,1)$, $(1,0)$ and $(1,-1)$. Accordingly the number of irreducible components is independent of the parameter ω and the same in every approximation. For classical Motzkin paths we have the additional requirement that the path is not allowed to cross the x-axis.

The approximations of the affine flag variety are compatible with its ind-variety structure in the sense that the local structure around the points is preserved along the embedding into a bigger approximation. Hence the ind-topology and the Zariski topology of the degenerate affine flag coincide.

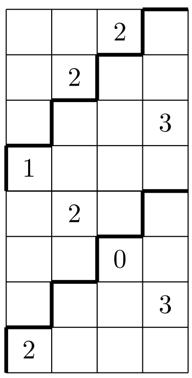
Ordinary Dellac configurations are in bijection with the cells in the degenerate full flag variety of type A. Affine Dellac configurations are a generalisation of these configurations to match the structure of the cells in the degenerate affine flag variety. Classically the configurations consist of marked and unmarked boxes in a rectangle of boxes. For the affine case we need some additional parameters to distinguish between the cells in the different approximations.

Definition 3. The set of affine Dellac configurations to the parameter ω is denoted by $\widehat{DC}_n(\omega)$. A configuration $\hat{D} \in \widehat{DC}_n(\omega)$ consists of a rectangle of $2n \times n$ boxes with $2n$ entries $k_j \in \{0, 1, 2, \ldots, \omega\}$ such that:

(1) There is one number in each row
(2) There are two numbers in each column
(3) $\sum_{j=1}^{2n} (p_j + nr_j) = \omega n^2$.

Here p_j is the number of steps from the separator to the entry going left and $r_j := \max\{k_j - 1, 0\}$. If the entry is zero then the position is zero as well. The left hand side and the right hand side of the rectangle are identified to obtain boxes on a cylinder. The separator is a staircase around the cylinder. In the planar picture we draw it from the lower left corner to the upper right corner of the rectangle of boxes.

Example 1. For $n = 4$ and $\omega = 3$ the subsequent configuration



	p_j :	r_j :
	1	1
	1	1
	2	2
	4	0
	2	1
	0	0
	2	2
	4	1
$\Sigma =$	16	8

is contained in the set $\widehat{DC}_4(3)$ since $16 + 4 \cdot 8 = 48 = 3 \cdot 4^2$.

There exists a function

$$\begin{aligned} h : \widehat{DC}_n(\omega) &\longrightarrow \mathbb{Z} \\ D &\longmapsto h(D) \end{aligned}$$

such that $h(D)$ is equal to the dimension of the corresponding cell in the approximation of the affine flag variety.

THEOREM 3 (Theorem 6.59). For $\omega \in \mathbb{N}$, the Poincaré polynomial of $\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ is given by

$$p_{\mathcal{F}l^a_\omega\left(\widehat{\mathfrak{gl}}_n\right)}(q) = \sum_{D \in \widehat{DC}_n(\omega)} q^{h(D)}.$$

For the linear degenerations of the affine flag variety as mentioned above we introduce an order depending on the co-ranks of the maps f_i to distinguish the degenerations between the affine flag variety and the Feigin degeneration from the other degenerations. For these intermediate degenerations we can define partial degenerate affine Dellac configurations to parametrise their cells (Theorem 6.55). Moreover there exist dimension functions for these configurations such that we obtain analogous descriptions of the Poincaré polynomials of the partial degenerate affine flag varieties (Theorem 6.61).

Methods and Structure

For the proofs of Theorem 1 and Theorem 2 it is necessary to understand the quiver Grassmannians for the oriented cycle and their geometric properties. In Chapter 1 we recall some basic results, constructions and definitions concerning quiver representations and quiver Grassmannians in general. We introduce the two different realisations of quiver Grassmannians which are both used at various points in this thesis. They arise from the different possibilities to describe a subrepresentation. We introduce the path algebra of a quiver and representations of quivers with relations. For an admissible set of relations there is an equivalence of bounded quiver representations and modules over finite dimensional algebras.

The link between representations of quivers and modules over finite dimensional algebras is the foundation of the realisation of quiver Grassmannians as framed module spaces which is proven in Chapter 2. This interpretation of quiver Grassmannians allows us to translate properties between the variety of quiver representations and the quiver Grassmannian. For the proof it is necessary that we restrict us to representations which consist only of injective summands and satisfy relations from an admissible set. In some cases it is easier to study the orbits in the first variety instead of strata in the quiver Grassmannian. The proof that this construction preserves the geometry is based on deframing of extended quiver representations and stability conditions for quiver representations.

The equioriented cycle and the class of quiver Grassmannians which we want to examine is introduced Chapter 3. Based on word combinatorics we prove a dimension formula for the space of morphisms between nilpotent indecomposable representations of the cycle. This is applied to parametrise the irreducible components and prove the geometric properties of the quiver Grassmannians for the cycle as claimed in Theorem 2 of this introduction. The proof of the geometric properties utilises the construction of the quiver Grassmannian as framed moduli space. Hence we can lift them from the variety of quiver representations which was studied by G. Kempken [**48**]. In Section 3.1 we summarise some results from her thesis which was only published in German. This is one main ingredient for the proof of Theorem 2.

In Chapter 4 we introduce a $\mathbb{C}^*$-action on the quiver Grassmannians for the equioriented cycle and recall some facts about $\mathbb{C}^*$-actions and decompositions. This action provides us a combinatoric tool to compute the Euler characteristic of these quiver Grassmannians which was introduced by G. Cerulli Irelli in [**16**]. It induces a cellular decomposition which allows us to compute the Poincaré polynomials of the quiver Grassmannians. For the proof of the decomposition it is crucial to find the right grading for the action on the indecomposable summands of the quiver representation which generalises to the action on the quiver Grassmannian.

The theory from the first chapters is applied in Chapter 5 to identify approximations for partial degenerations of the affine Grassmannian with quiver Grassmannians for the loop quiver. This generalises the construction by E. Feigin, M. Finkelberg and M. Reineke [**30**]. It is the foundation for the study of the partial degenerations of the affine flag. The loop is a special case of the cycle such that we can apply the theory from the previous chapters concerning the cellular decomposition and the geometric properties. We compute different parametrisations of the cells and give formulas for the Poincaré polynomials for the partial degenerations and their finite approximations.

The main part of this thesis is Chapter 6 where we identify finite approximations of partial degenerate affine flag varieties with quiver Grassmannians for the equioriented cycle. Based on the previous chapters we generalise the construction for the affine Grassmannian. This allows us to describe cellular decompositions via successor closed subquivers in the coefficient quiver of the quiver representation which corresponds to the approximation. These subquivers turn out to be parametrised by affine Dellac configurations. From the configurations it is possible to recover the dimension of the corresponding cell inducing a formula for the Poincaré polynomial of the approximations. Moreover the notion of affine Dellac configurations is compatible with the partial degenerations of the affine flag variety and the formula for the Poincaré polynomial generalises to this setting. The parametrisation of the irreducible components of the quiver Grassmannians for the cycle as computed in Chapter 3 allows to identify the irreducible components of the degenerate affine flag variety and its approximations with grand Motzkin paths. Together with the geometric properties which lift from the variety of quiver representations this finishes the proof of Theorem 2.

In Chapter 7 we recall the definition of a moment graph and its application to compute the equivariant intersection comomology of varieties with a suitable torus action. We introduce a combinatoric approach to compute the moment graph for quiver Grassmannians which is based on the $\mathbb{C}^*$-action and the induced cellular decomposition of the quiver Grassmannian. Some parts of this Chapter are still conjectural.

The class of quiver Grassmannians studied in this thesis has some rather strong restrictions. In Appendix A we give some examples to point out the various problems and difficulties which turn up if one leaves this class of quiver Grassmannians or tries to relax the length condition for the nilpotent indecomposable representations. Moreover we give some counter examples for properties which could still not be satisfied for the class of studied quiver Grassmannians.

The parametrisation of the cells via successor closed subquivers in the coefficient quiver of the representation describing the quiver Grassmannian as introduced in Chapter 4 allows determine the Poincaré polynomials computationally. The implementations of these programs are presented in Appendix B and some results of the computations are given in Appendix C.

Outlook

For the study of the approximations of the partial degenerate affine Grassmannians new methods are required because the indecomposable summands of the corresponding quiver representation do not have all the same length. In the framework of this thesis it was not possible to prove an explicit formula for their dimension and irreducible components or examine their geometric properties. The same is true for the partial degenerations of the affine flag variety.

The explicit computation of the equivariant intersection comomology of a quiver Grassmannian for the equioriented cycle or other quivers is still an open problem. It would be interesting to see in which generality the combinatoric construction of the moment graph is possible and if this induces some general description of the equivariant intersection comomology for quiver Grassmannians.

CHAPTER 1

Basic facts about Quiver Grassmannians

In this chapter we give a formal introduction to quiver Grassmannians and recall some results which we want to apply in later Chapters. Let $\Bbbk$ be an algebraically closed field of characteristic zero. For the study of the affine flag variety we will even restrict to the complex numbers. This introduction follows the article by G. Cerulli Irelli [**17**] and the reference therein. More detail and background information are provided in the book by I. Assem, D. Simson and A. Skowronski [**2**], the book by R. Schiffler [**66**], the lecture notes by O. Schiffmann [**67**] and the book by A. Kirillov Jr. [**50**].

1.1. Quiver Representations

A (finite) **quiver** $Q = (Q_0, Q_1, s, t)$ is an ordered quadruple where:

(1) Q_0 denotes a finite set of vertices,

(2) Q_1 is a finite set of edges,

(3) The functions $s, t : Q_1 \to Q_0$ provide an orientation of the edges.

For an oriented edge α we write $\alpha : s_\alpha \to t_\alpha$, i.e. the function s sends an edge to its **source** and the function t sends it to its **target**.

A (finite dimensional) **Q-representation** is a pair of tuples

$$M := \Big(\big(V_i\big)_{i\in Q_0}, \big(M_\alpha\big)_{\alpha\in Q_1} \Big) \quad \text{where :}$$

(1) V_i is a finite dimensional vector space over the field $\Bbbk$ for all $i \in Q_0$,

(2) $M_\alpha : V_{s_\alpha} \to V_{t_\alpha}$ is a linear map for all $\alpha \in Q_1$.

A **Q-morphism** $\psi : M \to N$ of two Q-representations is a collection of linear maps

$$\Big(\psi_i : V_i \to W_i \Big)_{i\in Q_0}$$

such that the following diagrams are commutative

$$\begin{array}{ccc} V_{s_\alpha} & \xrightarrow{\psi_{s_\alpha}} & W_{s_\alpha} \\ {\scriptstyle M_\alpha}\big\downarrow & /\!\!/\!\!/ & \big\downarrow{\scriptstyle N_\alpha} \\ V_{t_\alpha} & \xrightarrow[\psi_{t_\alpha}]{} & W_{t_\alpha} \end{array}$$

i.e. $\psi_{t_\alpha} \circ M_\alpha \equiv N_\alpha \circ \psi_{s_\alpha}$ holds for every arrow $\alpha \in Q_1$. With $\mathrm{Hom}_Q(M, N)$ we denote the set of all Q-morphisms from M to N. We call an injective Q-morphism $\iota : U \hookrightarrow M$ **embedding**.

The dimension vector of a Q-representation M is defined as

$$\mathbf{dim}\, M := \big(\dim V_i\big)_{i\in Q_0}.$$

In the introduction the **quiver Grassmannian** $\mathrm{Gr}^Q_{\mathbf{e}}(M)$ is defined as the set of all subrepresentations of the Q-representation M with dimension vector $\mathbf{e} \in \mathbb{Z}^{Q_0}$. There are two different interpretations of the notion subrepresentation. The first one as mentioned in the introduction parametrises a subrepresentation as tuple of subspaces $(U_i)_{i\in Q_0}$ in the vector spaces over the vertices of the quiver, i.e $U_i \subset M_i$ for all $i \in Q_0$. Additionally these tuples have to be compatible with maps between the vectorspaces of M, i.e.

$$M_\alpha\big(U_{s_\alpha}\big) \subseteq U_{t_\alpha}$$

for all oriented edges $\alpha \in Q_1$. In the above notation for Q-representations this type of subrepresentation is written as

$$U := \Big(\big(U_i\big)_{i\in Q_0}, \big(M_\alpha|_{U_{s_\alpha}}\big)_{\alpha\in Q_1} \Big)$$

and the embedding ι is given by the identity map.

In general a **subrepresentation** of M is a tuple (U, φ) consisting of a Q-representation U and a embedding $\varphi : U \hookrightarrow M$ such that the diagrams above commute and all component mas φ_i are injective. These different interpretations lead to the two different realisations of the quiver Grassmannian recalled below.

1.2. Universal Quiver Grassmannians

In this section we construct quiver Grassmannians based on the subspace parametrisation of subrepresentations. Let $\mathrm{rep}(Q)$ be the category of finite dimensional Q-representations. For a fixed dimension vector $\mathbf{d} \in \mathbb{Z}^{Q_0}_{\geq 0}$ we denote by $\mathrm{rep}_{\mathbf{d}}(Q)$ the category of Q-representations with dimension vector $\mathbf{d}$.

The objects of the category $\mathrm{rep}_{\mathbf{d}}(Q)$ are parametrised by the variety

$$\mathrm{R}_{\mathbf{d}}(Q) := \bigoplus_{\alpha\in Q_1} \mathrm{Hom}_{\Bbbk}(\Bbbk^{d_{s_\alpha}}, \Bbbk^{d_{t_\alpha}})$$

which is called the **variety of quiver representations** for the dimension vector $\mathbf{d}$. This means that for a fixed dimension vector every Q-representation is determined by the maps along the oriented edges of the quiver up to base change of the vector spaces over the vertices. The group

$$\mathrm{GL}_{\mathbf{d}} := \prod_{i\in Q_0} \mathrm{GL}_{d_i}(\Bbbk)$$

acts on the points M in this variety via base change, i.e.

$$g.M := \Big(g_{t_\alpha} M_\alpha g^{-1}_{s_\alpha} \Big)_{\alpha\in Q_1}.$$

The dimension of an orbit $\mathcal{O}_M := \mathrm{GL}_{\mathbf{d}}.M$ is given as

$$\dim \mathcal{O}_M = \dim \mathrm{GL}_{\mathbf{d}} - \dim \mathrm{End}_Q\big(M\big) = \sum_{i\in Q_0} d_i^2 - \dim \mathrm{Hom}_Q\big(M, M\big)$$

and the isomorphism classes of quiver representations and the $\mathrm{GL}_{\mathbf{d}}$-orbits coincide.

Now let $\mathbf{e}$ and $\mathbf{d}$ be two dimension vectors in $\mathbb{Z}^{Q_0}$ such that $0 \leq e_i \leq d_i$ holds for all $i \in Q_0$. We define the product of usual Grassmannians as

$$\mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) := \prod_{i \in Q_0} \mathrm{Gr}_{e_i}(\Bbbk^{d_i}).$$

The **universal quiver Grassmannian** is defined as

$$\mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d}) := \Big\{ (N, M) \in \mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) \times \mathrm{R}_{\mathbf{d}}(Q) : M_\alpha(N_{s_\alpha}) \subseteq N_{t_\alpha} \text{ for all } \alpha \in Q_1 \Big\}.$$

From the universal quiver Grassmannian we have two projections to the different components of its elements

$$\begin{array}{ccc} & \mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d}) & \\ {}^{\mathrm{p}_{\mathbf{e}}}\swarrow & & \searrow^{\mathrm{p}_{\mathbf{d}}} \\ \mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) & & \mathrm{R}_{\mathbf{d}}(Q) \end{array}$$

Finally, for a quiver Q, a dimension vector $\mathbf{e}$ and a representation $M \in \mathrm{R}_{\mathbf{d}}(Q)$ we obtain the corresponding quiver Grassmannian as

$$\mathrm{Gr}^Q_{\mathbf{e}}(M) := \mathrm{p}_{\mathbf{d}}^{-1}(M).$$

1.3. Quotient Construction of Quiver Grassmannians

In this section we construct quiver Grassmannians arising from the parametrisation of subrepresentations of a representation M as pair of a quiver representation N and an embedding $\varphi : N \hookrightarrow M$. For two dimension vectors $\mathbf{e}, \mathbf{d} \in \mathbb{Z}^{Q_0}_{\geq 0}$ define the variety of $\Bbbk$-morphisms

$$\mathrm{Hom}_{\Bbbk}\big(\mathbf{e}, \mathbf{d}\big) := \bigoplus_{i \in Q_0} \mathrm{Hom}_{\Bbbk}\big(\Bbbk^{e_i}, \Bbbk^{d_i}\big)$$

and use triples

$$\big(N, \varphi, M\big) := \Big(\big(N_\alpha\big)_{\alpha \in Q_1}, \big(\varphi_i\big)_{i \in Q_0}, \big(M_\alpha\big)_{\alpha \in Q_1} \Big) \in \mathrm{R}_{\mathbf{e}}(Q) \times \mathrm{Hom}_{\Bbbk}\big(\mathbf{e}, \mathbf{d}\big) \times \mathrm{R}_{\mathbf{d}}(Q)$$

to define the universal Q-morphism variety

$$\mathrm{Hom}_Q\big(\mathbf{e}, \mathbf{d}\big) := \Big\{ \big(N, \varphi, M\big) : \varphi_{t_\alpha} N_\alpha = M_\alpha \varphi_{s_\alpha} \text{ for all } \alpha \in Q_1 \Big\}.$$

Since we are interested in subrepresentations we have to restrict it to injective Q-morphisms

$$\mathrm{Hom}^0_Q\big(\mathbf{e}, \mathbf{d}\big) := \Big\{ \big(N, \varphi, M\big) \in \mathrm{Hom}_Q\big(\mathbf{e}, \mathbf{d}\big) \;\Big|\; \varphi_i : \Bbbk^{e_i} \hookrightarrow \Bbbk^{d_i} \text{ is inj. for all } i \in Q_0 \Big\}.$$

Similar as for the universal quiver Grassmannian we have projections from the variety of universal Q-morphisms to its components and analogous for the variety of injective Q-morphisms

$$\begin{array}{ccc} & \mathrm{Hom}_Q\big(\mathbf{e}, \mathbf{d}\big) & \\ {}^{\mathrm{pr}_{\mathbf{e}}}\swarrow & & \searrow^{\mathrm{pr}_{\mathbf{d}}} \\ \mathrm{R}_{\mathbf{e}}(Q) & & \mathrm{R}_{\mathbf{d}}(Q) \end{array} \qquad \begin{array}{ccc} & \mathrm{Hom}^0_Q\big(\mathbf{e}, \mathbf{d}\big) & \\ {}^{\widetilde{\mathrm{pr}}_{\mathbf{e}}}\swarrow & & \searrow^{\widetilde{\mathrm{pr}}_{\mathbf{d}}} \\ \mathrm{R}_{\mathbf{e}}(Q) & & \mathrm{R}_{\mathbf{d}}(Q) \end{array}$$

The universal quiver Grassmannian is the quotient of the variety of universal injective Q-morphisms by the group acting on the first component

$$\mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d}) \cong \mathrm{Hom}^0_Q(\mathbf{e}, \mathbf{d}) / \mathrm{GL}_{\mathbf{e}}.$$

For a fixed representation M we define

$$\mathrm{Hom}_Q^0(\mathbf{e}, M) := \widetilde{\mathrm{pr}}_{\mathbf{d}}^{-1}(M)$$

and obtain an isomorphic description of the quiver Grassmannian as the following quotient

$$\mathrm{Gr}_{\mathbf{e}}^Q(M) \cong \mathrm{Hom}_Q^0(\mathbf{e}, M)/\mathrm{GL}_{\mathbf{e}}.$$

This isomorphism is proven in Proposition 2.5.

1.4. Bound Quiver Representations and the Path Algebra

In the previous sections everything works in the setting of finite dimensional representations of a finite quiver. For the class of quiver Grassmannians for the equioriented cycle which is studied in this thesis we can restrict this generality to derive stronger statements about their geometry. Namely we want to work in the setting of finite dimensional modules over finite dimensional algebras. In this section we describe how this is related to representations of a finite quiver.

A **path** p in a quiver Q is a sequence of arrows $\alpha_1, \dots, \alpha_r \in Q_1$ such that $t_{\alpha_i} = s_{\alpha_{i+1}}$ holds for all $i \in [r-1] := \{1, 2, \dots, r-1\}$ and we write

$$p = (i|\alpha_1 \dots \alpha_r|j)$$

where $i = s_{\alpha_1}$ and $j = t_{\alpha_r}$. Using the same notation as for edges, the source of a path is defined as $s_p := s_{\alpha_1}$ and $t_p := t_{\alpha_r}$ denotes its target. Let p be a path with notation as above and $M \in \mathrm{R}_{\mathbf{e}}(Q)$ be a quiver representation. The map M_p is defined as

$$M_p := M_{\alpha_r} \circ M_{\alpha_{r-1}} \circ \dots \circ M_{\alpha_1}.$$

For two paths $p = (i|\alpha_1 \dots \alpha_r|j)$ and $p' = (k|\alpha'_1 \dots \alpha'_r|\ell)$ the concatenation

$$pp' := (i|\alpha_1 \dots \alpha_r \alpha'_1 \dots \alpha'_r|\ell)$$

is defined if $j = k$.

Definition 1.1. The **path algebra** $\Bbbk Q$ of a quiver Q is the $\Bbbk$-algebra with basis consisting of all paths in Q and multiplication of two paths p and p' defined as

$$p \cdot p' := \begin{cases} pp' & \text{if } t_p = s_{p'} \\ 0 & \text{otherwise.} \end{cases}$$

We call two paths p and p' in Q **parallel** if $s_p = s_{p'}$ and $t_p = t_{p'}$. For parallel paths of length greater than one, a **relation** ρ is a linear combination

$$\rho := \sum_p \lambda_p p$$

where $\lambda_p \in \Bbbk$ for all paths in the sum. The pair (Q, R) consisting of a quiver Q together with a set of relations R defines a **bound quiver**.

A representation of (Q, R) is a representation $M = ((V_i)_{i \in Q_0}, (M_\alpha)_{\alpha \in Q_1})$ of Q such that

$$M_\rho := \sum_p \lambda_p M_p \equiv 0$$

holds for all relations $\rho = \sum_p \lambda_p p$ in R. We call M **bound quiver representation**. For a set of relations R let $I :=< R >$ be the ideal in the path algebra $\Bbbk Q$ which is generated by the relations in R. If a representation M satisfies the relations in R it also satisfies all relations in the ideal I.

From now on we will write I for an ideal generated by some relations R and call the pair (Q, I) a bound quiver as well. For an ideal I with relations ρ, we define the **variety of bound quiver representations** as

$$\mathrm{R}_{\mathbf{d}}(Q,I) := \big\{ M \in \mathrm{R}_{\mathbf{d}}(Q) : M_\rho \equiv 0 \text{ for all } \rho \in I \big\}.$$

All the constructions of the previous sections can be done in the same way for bound quiver representations. A subrepresentation of a representation $M \in \mathrm{R}_{\mathbf{d}}(Q,I)$ satisfies the same relations as M. For this reason it is no restriction to assume the boundedness of the candidates for the subrepresentations in the universal quiver Grassmannian and the quotient construction of the quiver Grassmannian.

The arrow ideal R_Q of the path algebra $A := \Bbbk Q$ is the two-sided ideal generated by all arrows in Q. It admits the vector space decomposition

$$R_Q = \bigoplus_{\ell \geq 1} \Bbbk Q_\ell$$

where $\Bbbk Q_\ell$ is the vector subspace of $\Bbbk Q$ with paths of length ℓ as basis. The vector space decomposition of A runs over $\ell \geq 0$. For the k-th power of R_Q we have the decomposition

$$R_Q^k = \bigoplus_{\ell \geq k} \Bbbk Q_\ell$$

and its vector space basis consists of paths of length greater or equal to k.

A two-sided ideal I of A is called **admissible ideal** if there exists an integer $k \geq 2$ such that

$$R_Q^k \subset I \subset R_Q^2.$$

For an admissible I the pair (Q, I) defines a **bound quiver** and the quotient $\Bbbk Q/I$ is called **bound quiver algebra** (or bounded path algebra). The subsequent theorem connects the study of modules over finite dimensional algebras and bounded representations of finite quivers. It is proven in the book by R. Schiffler [**66**, Theorem 5.4].

THEOREM 1.2. *Let $A = \Bbbk Q/I$ be a bound quiver algebra of a finite connected quiver Q. Then there is an equivalence of categories between the category* A-mod *of finitely generated right A-modules and the category* $\mathrm{rep}_\Bbbk(Q, I)$ *of finite dimensional bound quiver representations.*

REMARK. Every basic finite dimensional $\Bbbk$-algebra is isomorphic to a bound quiver algebra for a finite connected quiver Q and some admissible ideal I.

We call a quiver representation **indecomposable** if it can not be written as the direct sum of two proper subrepresentations. A quiver representation is called **simple** if all maps along the arrows are zero, one vector space over the vertices is isomorphic to $\Bbbk$ and all other vector spaces are zero. Simple quiver representations are indecomposable. Every finite dimensional quiver representation has a decomposition into indecomposable representations which is unique up to the order of the summands [**50**, Theorem 1.11]. Hence it is sufficient to understand the indecomposable representations of a quiver. It is shown by P. Gabriel in [**35**] that a quiver admits a finite number of indecomposable representations if and only if it is a Dynkin quiver. This are quivers where the underlying graph (Q_0, Q_1) is a simply-laced Dynkin diagram [**66**, p. 83].

Moreover for Dynkin quivers the path algebra is already finite. Certain quiver Grassmannians for Dynkin quivers admit nice geometric properties like cellular decompositions into attracting sets of torus fixed points, irreducibility and normality as proven by G. Cerulli Irelli, E. Feigin and M. Reineke in [**20**]. But the cycle is unfortunately not a Dynkin quiver such that we can not apply their results for the study of the affine flag variety. Hence we have to develop methods which work in a bigger generality. It is the setting of finite dimensional modules over finite dimensional algebras which we choose. In Chapter 2 we will examine quiver Grassmannians for quiver representations which are the direct sum of injective representations of a bound quiver (Q, I).

The indecomposable **projective representation** P_i of the bound quiver (Q, I) is given by the pair of tuples

$$P_i = \Big(\big(P_j^{(i)}\big)_{j \in Q_0}, \big(P_\alpha^{(i)}\big)_{\alpha \in Q_1} \Big)$$

where $P_j^{(i)}$ has a basis of equivalence classes $\overline{p}$ of non-constant paths from i to j in Q and for an arrow $\alpha : j \to \ell$ in Q the map

$$P_\alpha^{(i)} : P_j^{(i)} \to P_\ell^{(i)}$$

is defined on the basis consisting of paths by the composition of paths from i to j with the arrow α, i.e.

$$P_\alpha^{(i)}(\overline{p}) = \overline{\alpha p}.$$

The indecomposable **injective representation** I_i of the bound quiver (Q, I) is given by the pair of tuples

$$I_i = \Big(\big(I_j^{(i)}\big)_{j \in Q_0}, \big(I_\alpha^{(i)}\big)_{\alpha \in Q_1} \Big)$$

where $I_j^{(i)}$ has a basis of equivalence classes $\overline{p}$ of non-constant paths from j to i in Q and for an arrow $\alpha : j \to \ell$ in Q the map

$$I_\alpha^{(i)} : I_j^{(i)} \to I_\ell^{(i)}$$

is defined on the basis consisting of paths by deleting the arrow α from paths going from j to i, i.e.

$$I_\alpha^{(i)}(\overline{p}) = \begin{cases} \overline{p'} & \text{if } p = \alpha p' \\ 0 & \text{otherwise.} \end{cases}$$

Using projective and injective we can interpret the path algebra and its dual as

$$A = \bigoplus_{i \in Q_0} P_i \quad \text{and} \quad A^* = \bigoplus_{j \in Q_0} I_j.$$

In Chapter 2 we will examine quiver Grassmannians for quiver representations which are the direct sum of injective representations of a bound quiver (Q, I). It turns out that for the oriented cycle this class of quiver Grassmannians is similar to the class of quiver Grassmannians which are studied in [**20**].

1.5. Stratification of Quiver Grassmannians

For the variety of quiver representations its orbit structure has been studied by many authors, see for example [**1, 8, 9, 48**]. The analogous structure for quiver Grassmannians is their stratification. In Chapter 2 we study the connection between the stratification of quiver Grassmannians and the orbit structure of the variety of quiver representations.

The **stratum** $\mathcal{S}_N$ of a subrepresentation $N \in \mathrm{Gr}_{\mathbf{e}}^Q(M)$ is the set of all subrepresentations $U \in \mathrm{Gr}_{\mathbf{e}}^Q(M)$ isomorphic to N, namely

$$\mathcal{S}_N = \{V \in \mathrm{Gr}_{\mathbf{e}}^Q(M) : V \cong N\}.$$

We can use the quotient construction of the quiver Grassmannian to give a more formal definition. Restricting the projection to the first component $\widetilde{\mathrm{pr}}_{\mathbf{e}}$ to the pre-image of the representation M we obtain the following map

$$\mathrm{p}_0 : \mathrm{Hom}_Q^0(\mathbf{e}, M) \to \mathrm{R}_{\mathbf{e}}(Q)$$

and can redefine the stratum of N as

$$\mathcal{S}_N \cong \mathrm{p}_0^{-1}\big(\mathrm{GL}_{\mathbf{e}}.N\big)/\mathrm{GL}_{\mathbf{e}}.$$

For a Q-representation with finitely many isomorphism classes of subrepresentations the quiver Grassmannian admits a finite stratification

$$\mathrm{Gr}_{\mathbf{e}}^Q(M) = \coprod \mathcal{S}_N.$$

This is for example true for representations of Dynkin quivers. Moreover we have the subsequent formula for the dimensions of the strata. For Dynkin quivers this statement is proven by G. Cerulli Irelli, E. Feigin and M. Reineke in the article [**20**, Lemma 2.4].

LEMMA 1.3. *Each $\mathcal{S}_N$ is an irreducible locally closed subset of $\mathrm{Gr}_{\mathbf{e}}^Q(M)$ of dimension*

$$\dim \mathrm{Hom}_Q(N, M) - \dim \mathrm{End}_Q(N).$$

With the same methods it is possible to prove this statement in the setting of modules over finite dimensional algebras. That is the generality which is required to apply it to the representations of the equioriented cycle as introduced in Chapter 3.

CHAPTER 2

Framed Moduli Interpretation

In this chapter we introduce one of the main tools used in this thesis for the study of quiver Grassmannians. It allows us to translate geometric properties from the variety of quiver representations to the quiver Grassmannian for a certain class of quiver representations M and dimension vectors $\mathbf{e}$. In our setting the variety of quiver representations has been studied intensively by G. Kempken in [**48**] whereas about the corresponding quiver Grassmannians there is not a lot know. Some special cases of quiver Grassmannians for the loop quiver were studied by N. Haupt in [**41**]. The method to lift geometric properties from the variety of quiver representations to the quiver Grassmannian was already known to K. Bongartz [**10**] before it was proven by M. Reineke for Dynkin quivers in [**62, 63**]. Some parts of the proof were generalised to the setting of finite dimensional algebras by S. Fedotov in [**27**]. In the remainder of this chapter we generalise the statement and its proof to the setting of finite dimensional algebras.

The **extended representation variety** is defined as

$$\mathrm{R}_{\mathbf{e},\mathbf{d}}(Q,I) := \mathrm{R}_{\mathbf{e}}(Q,I) \times \mathrm{Hom}_{\Bbbk}(\mathbf{e},\mathbf{d})$$

Definition 2.1. A point (M,f) of $\mathrm{R}_{\mathbf{e},\mathbf{d}}(Q,I)$ is called **stable** if there is no non-zero subrepresentation U of M which is contained in $\mathrm{Ker} f \subseteq M$. The set of all stable points of $\mathrm{R}_{\mathbf{e},\mathbf{d}}(Q,I)$ is denoted by $\mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I)$.

Theorem 2.2. *Let Q be a finite connected quiver and I an admissible ideal of the path algebra $\Bbbk Q$. The indecomposable injective representation of the bound quiver (Q,I) ending at vertex $j \in Q_0$ is denoted by I_j. Then*

$$\mathrm{Gr}^Q_{\mathbf{e}}(J) \quad \cong \quad \mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q,I)$$

where

$$J := \bigoplus_{j\in Q_0} I_j \otimes \Bbbk^{d_j}$$

and $\mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q,I)$ is the geometric quotient of $\mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I)$ by the group $\mathrm{GL}_{\mathbf{e}}$.

Deviating from the previous chapter $\mathbf{d}$ is a tuple with multiplicities of injective bounded quiver representations and not the dimension vector of the quiver representation J. The proof of this theorem is given in Section 2.1. This theorem was fist proven by M. Reineke for Dynkin quivers in [**63**, Proposition 3.9]. S. Fedotov used the same methods to derive the statement in the generality of modules over finite dimensional algebras in [**27**, Theorem 3.5].

The subsequent theorem establishes a bijection between orbits in the variety of quiver representations and strata in the corresponding quiver Grassmannian which preserves geometric properties. It allows us to lift the properties of the variety of quiver representation studied by G. Kempken to the quiver Grassmannians which are used for the finite approximations of the affine flag variety and its degenerations in Chapter 6. In the case of Dynkin quivers it is proven by M. Reineke in [**63**, Theorem 6.4].

Define $\mathrm{R}_{\mathbf{e}}^{(\mathbf{d})}(Q,I)$ as the image of the projection

$$pr : \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I) \to \mathrm{R}_{\mathbf{e}}(Q,I).$$

Theorem 2.3. *There is a bijection between $\mathrm{Aut}_Q(J)$-stable subvarieties of $\mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q,I)$ and $\mathrm{GL}_{\mathbf{e}}$-stable subvarieties of $\mathrm{R}_{\mathbf{e}}^{(\mathbf{d})}(Q,I)$ such that inclusions, closures, irreducibility and types of singularities are preserved.*

We postpone the proof to Section 2.5 until we established the framed module interpretation of the quiver Grassmannians.

2.1. Quotient Construction and Framed Moduli Spaces

In this section we prove Theorem 2.2 following the approach by M. Reineke [**63**]. For the proof we use the quotient construction of quiver Grassmannians by P. Caldero and M. Reineke [**15**, Lemma 2]. They proved it in the setting of quivers without oriented cycles. Below we generalise their proof to the setting of arbitrary finite quivers. Based on the quotient construction of the quiver Grassmannian the idea of the proof for Theorem 2.2 is to define a map

$$\Phi : \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I) \to \mathrm{Hom}^0_Q(\mathbf{e},J)$$

which is bijective and descends to an isomorphism of the $\mathrm{GL}_{\mathbf{e}}$-quotients. In the first part of this section we prove the quotient construction and introduce a map whose properties are examined in the second part of the section.

Definition 2.4. Let G be an algebraic group and X a G-variety. A geometric quotient of X by G is a pair (Y,π) with a morphism $\pi : X \to Y$ satisfying the properties:

(1) π is surjective and its fibres are exactly the G-orbits in X.
(2) A subset $U \subset Y$ is open if and only if $\pi^{-1}(U) \subset X$ is open.
(3) The sheaves $\mathcal{O}_Y$ and $(\pi_*\mathcal{O}_X)^G$ are equal on Y.

By (1) and (2) we can identify Y with the orbit space X/G and the variety structure of Y is uniquely determined by (3). For an introduction to actions of algebraic groups we refer to the lecture notes by M. Brion [**13**]. Quiver Grassmannians for finite quivers admit a description as certain geometric quotients.

Proposition 2.5. Let M be a representation of the finite quiver Q and $\mathbf{e}$ a dimension vector with $0 \leq \mathbf{e} \leq \mathbf{dim}\, M$. The quiver Grassmannian

$$\mathrm{Gr}^Q_{\mathbf{e}}(M)$$

isomorphic to the geometric quotient

$$\mathrm{Hom}^0_Q(\mathbf{e},M)/\mathrm{GL}_{\mathbf{e}}.$$

Proof. Define the map

$$\Phi : \mathrm{Hom}^0_Q(\mathbf{e},M) \to \mathrm{Gr}^Q_{\mathbf{e}}(M)$$

$$(N,\psi) \mapsto \big(\psi(N), M|_{\psi(N)}\big) := \Big(\big(\psi_i(\Bbbk^{e_i})\big)_{i\in Q_0}, \big(M_\alpha\big|_{\psi(N)}\big)_{\alpha\in Q_1}\Big)$$

where $N_i = \Bbbk^{e_i}$ since $N \in \mathrm{R}_{\mathbf{e}}(Q)$. The map ψ is a Q-morphism and hence satisfies

$$\psi_{t_\alpha} N_\alpha \equiv M_\alpha \psi_{s_\alpha} \text{ for all } \alpha \in Q_1.$$

This implies that

$$M_\alpha|_{\psi(N)}\big(\psi_{s_\alpha}(\Bbbk^{e_{s_\alpha}})\big) \equiv \psi_{t_\alpha}\big(N_\alpha(\Bbbk^{e_{s_\alpha}})\big) \text{ for all } \alpha \in Q_1.$$

Since N is a Q-representation it satisfies

$$N_\alpha(\Bbbk^{e_{s_\alpha}}) \subseteq \Bbbk^{e_{t_\alpha}} \text{ for all } \alpha \in Q_1$$

which implies

$$M_\alpha|_{\psi(N)}\big(\psi_{s_\alpha}(\Bbbk^{e_{s_\alpha}})\big) \subseteq \psi_{t_\alpha}\big(\Bbbk^{e_{t_\alpha}}\big) \text{ for all } \alpha \in Q_1$$

because ψ is injective and linear. Hence the tuple of vector spaces $\psi(N)$ describes a subrepresentation of M such that the pair $\big(\psi(N), M|_{\psi(N)}\big)$ is indeed included in the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^Q(M)$.

Let

$$V := \Big(\big(V_i\big)_{i\in Q_0}, \big(M_\alpha|_{V_{s_\alpha}}\big)_{\alpha\in Q_1}\Big)$$

be an element of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^Q(M)$ which is written in terms of the interpretation of the quiver Grassmannian via the universal Grassmannian. The inclusions $V_i \subseteq \Bbbk^{d_i}$ for all $i \in Q_0$ can be described by injective linear maps

$$f_i : \Bbbk^{e_i} \to \Bbbk^{d_i} \text{ such that } V_i = f_i\big(\Bbbk^{e_i}\big).$$

Define the pair (N, f) as

$$(N, f) := \Big(\big(N_\alpha\big)_{\alpha\in Q_1}, \big(f_i\big)_{i\in Q_0}\Big)$$

where

$$N_\alpha := f_{t_\alpha}|_{V_{t_\alpha}} \circ M_\alpha|_{V_{s_\alpha}} \circ \Big(f_{s_\alpha}|_{V_{s_\alpha}}\Big)^{-1}.$$

Then $N \in \mathrm{R}_{\mathbf{e}}(Q)$ holds because f is injective and linear and $M \in \mathrm{R}_{\mathbf{e}}(Q)$. If M satisfies bounding relations from an admissible ideal I the same relations are satisfied by N. The pair (N, f) we defined above is included in the space of injective Q-morphisms $\mathrm{Hom}_Q^0(\mathbf{e}, M)$ by construction. Its image under the map Φ is V which proves that Φ is surjective. Now let $(N, \psi), (U, \varphi) \in \mathrm{Hom}_Q^0(\mathbf{e}, M)$ be given such that $\Phi(N, \psi) = \Phi(U, \varphi)$. For the first component of the images this implies the equalities

$$\psi_i\big(\Bbbk^{e_i}\big) = \varphi_i\big(\Bbbk^{e_i}\big) \text{ for all } i \in Q_0.$$

Here ψ_i and φ_i are injective linear maps with the same image such that we can define

$$g_i := \Big(\psi_i\big|_{\mathrm{Im}\,\psi_i}\Big)^{-1} \circ \varphi_i\big|_{\mathrm{Im}\,\varphi_i} \text{ and } h_i := \Big(\varphi_i\big|_{\mathrm{Im}\,\varphi_i}\Big)^{-1} \circ \psi_i\big|_{\mathrm{Im}\,\psi_i} \in \mathrm{End}_\Bbbk\big(\Bbbk^{e_i}\big)$$

and obtain $h_i = g_i^{-1}$ and $g_i \in \mathrm{GL}_{e_i}(\Bbbk) = \mathrm{Aut}_\Bbbk\big(\Bbbk^{e_i}\big)$. Hence the tuple of the g_i's is included in the group $\mathrm{GL}_{\mathbf{e}}$ and we obtain

$$\Phi(N, \psi) = \Phi(g.U, g.\varphi).$$

So (N, ψ) and (U, φ) have to live in the same $\mathrm{GL}_{\mathbf{e}}$-orbit if they have the same image under the map Φ.

Conversely we compute $\Phi(g.U, g.\varphi)$ for $(U,\varphi) \in \mathrm{Hom}^0_Q(\mathbf{e}, M)$ and $g \in \mathrm{GL}_\mathbf{e}$. Using the formula for the action of $\mathrm{GL}_\mathbf{e}$ defined above we obtain

$$\begin{aligned}
\Phi(g.U, g.\varphi) &= \Big(g.\varphi\big(g.U\big), M|_{g.\varphi(g.U)}\Big)\\
&= \Big(\big(\varphi_i g_i^{-1}(\mathbb{k}^{e_i})\big)_{i\in Q_0}, \big(M_\alpha|_{g.\varphi_{s_\alpha}(g.U)}\big)_{\alpha\in Q_1}\Big)\\
&= \Big(\big(\varphi_i(\mathbb{k}^{e_i})\big)_{i\in Q_0}, \big(M_\alpha|_{\varphi_{s_\alpha} g_{s_\alpha}^{-1}(\mathbb{k}^{e_{s_\alpha}})}\big)_{\alpha\in Q_1}\Big)\\
&= \Big(\big(\varphi_i(\mathbb{k}^{e_i})\big)_{i\in Q_0}, \big(M_\alpha|_{\varphi_{s_\alpha}(\mathbb{k}^{e_{s_\alpha}})}\big)_{\alpha\in Q_1}\Big) = \Phi(U,\varphi)
\end{aligned}$$

which proves that Φ is constant on $\mathrm{GL}_\mathbf{e}$-orbits. Let $V \in \mathrm{Gr}^Q_\mathbf{e}(M)$ and $N \in \mathrm{Hom}^0_Q(\mathbf{e}, M)$ constructed from V as above. Then

$$\Phi^{-1}(V) = \mathrm{GL}_\mathbf{e}.N \text{ for all } V \in \mathrm{Gr}^Q_\mathbf{e}(M)$$

which implies the isomorphism. □

Let π be the quotient map

$$\begin{aligned}
\pi : \mathbb{k}Q &\to \mathbb{k}Q/I\\
p &\mapsto \overline{p}
\end{aligned}$$

where $\overline{p}$ denotes the set of all elements of the path algebra which are equivalent to p, i.e.

$$\overline{p} := \big\{q \in \mathbb{k}Q : \text{there exists an } r \in I \text{ such that } q = p + r\big\}.$$

The bounded path algebra $\mathbb{k}Q/I$ is finite dimensional since the ideal I is admissible. The condition that I is admissible implies that there exists an integer m such that all paths consisting of more than m arrows are included in the ideal I. Let $B(Q,I)$ denote a $\mathbb{k}$-basis of $\mathbb{k}Q/I$. For $i,j \in Q_0$ let $P_{i,j}(Q,I)$ be the set of paths from i to j in the bounded path algebra, i.e.

$$P_{i,j}(Q,I) := \big\{p \in \mathbb{k}Q : s_p = i, t_p = j \text{ and } p \notin I\big\}.$$

Here the condition $p \notin I$ ensures that for $p, q \in P_{i,j}(Q,I)$ with $p \neq q$ we obtain $\overline{p} \neq \overline{q}$. The sets $P_{i,j}(Q,I)$ are finite for all $i,j \in Q_0$ since the ideal I is admissible and hence contains all paths which are longer than a fixed integer m. In the sets $P_{i,i}(Q,I)$ we also have the constant path ϵ_i over the vertex $i \in Q_0$. The sum of all constant paths is the identity element of the path algebra $\mathbb{k}Q$.

Proposition 2.6. A $\mathbb{k}$-basis $B(Q,I)$ of the bounded path algebra $\mathbb{k}Q/I$ is given by

$$B(Q,I) := \pi\big(P(Q,I)\big) := \bigcup_{i,j\in Q_0} \pi\Big(\,P_{i,j}(Q,I)\,\Big).$$

Proof. The map $\pi : P(Q,I) \to B(Q,I)$ is injective by the definition of the sets $P_{i,j}(Q,I)$. For any $\overline{p} \in \mathbb{k}Q/I$ we can choose a representative p which is not included in I. Hence p has to be included in one of the sets $P_{i,j}(Q,I)$ and it is clear that the set $B(Q,I)$ generates the bounded path algebra as $\mathbb{k}$-vector space. The generating system $B(Q,I)$ is minimal since the image $\overline{p}$ of the path $p \notin I$ can not be written in terms of the generators $\pi\big(P(Q,I) \setminus \{p\}\big)$. This holds for any $p \in P_{i,j}(Q,I)$ by the definition of these sets. □

Using the sets of paths in the bounded path algebra we can define a map from the variety of framed quiver representations to the variety of Q-morphisms as follows.

DEFINITION 2.7. For an admissible ideal I of the finite quiver Q and a injective bounded Q-representation J we define the map

$$\Phi : \mathrm{R}_{\mathbf{e},\mathbf{d}}(Q,I) \to \mathrm{Hom}_Q(\mathbf{e},J)$$
$$(M,f) \mapsto \big(M, \varphi_{(M,f)}\big)$$

where $\mathbf{n} := \mathbf{dim}\, J$ and the components of $\varphi_{(M,f)}$ are given as

$$\varphi^{(i)}_{(M,f)} := \varphi_i := \bigoplus_{j\in Q_0} \bigoplus_{p\in P_{i,j}(Q,I)} f_j \circ M_p : \mathbb{k}^{e_i} \longrightarrow \mathbb{k}^{n_i} = \bigoplus_{j\in Q_0} \bigoplus_{p\in P_{i,j}(Q,I)} \mathbb{k}^{d_j}$$

for all $i \in Q_0$.

Here M_p is the concatenation of the maps M_α for the edges α which build the path p as defined in Section 1.4 and $f_j \in \mathrm{Hom}_{\mathbb{k}}(\mathbb{k}^{e_j}, \mathbb{k}^{d_j})$. The representation J is injective. Hence we obtain

$$\sum_{j\in Q_0} \sum_{p\in P_{i,j}(Q,I)} d_j = n_i = \big(\mathbf{dim}\, J\big)_i$$

and $\varphi_i \in \mathrm{Hom}_{\mathbb{k}}(\mathbb{k}^{e_i}, \mathbb{k}^{n_i})$ for all $i \in Q_0$.

In order to show that Φ is well defined we have to check that

$$\varphi^{(t_\alpha)}_{(M,f)} \circ M_\alpha \equiv J_\alpha \circ \varphi^{(s_\alpha)}_{(M,f)}$$

holds for all pairs $(M,f) \in \mathrm{R}_{\mathbf{e},\mathbf{d}}(Q,I)$ and all $\alpha \in Q_1$. Form the definition of $\varphi^{(t_\alpha)}_{(M,f)}$ we obtain that the map on the left is given by

$$\left(\bigoplus_{j\in Q_0} \bigoplus_{p\in P_{t_\alpha,j}(Q,I)} f_j \circ M_p \right) \circ M_\alpha .$$

On the right hand side we have

$$J_\alpha \circ \left(\bigoplus_{j\in Q_0} \bigoplus_{p\in P_{s_\alpha,j}(Q,I)} f_j \circ M_p \right).$$

The representation J is the sum of injective representations I_j, i.e.

$$J = \bigoplus_{j\in Q_0} I_j \otimes \mathbb{k}^{d_j}.$$

The vector space $I^{(j)}_i$ over the vertex $i \in Q_0$ belonging to the injective bounded representation I_j has a basis labelled by equivalence classes of paths from j to i in Q and the map $I^{(j)}_\alpha$ is the projection sending all basis vectors to zero whose indexing path is not going through α. The other basis elements are send to those whose indexing path is obtained by removing the edge α. Accordingly J_α is acting in the same way and the right hand side is equal to

$$\bigoplus_{j\in Q_0} \bigoplus_{p\in P_{t_\alpha,j}(Q,I)} f_j \circ M_p \circ M_\alpha .$$

This proves $\varphi_{(M,f)} \in \mathrm{Hom}_Q(M,J)$ and that the map Φ is well defined.

LEMMA 2.8. *The subspace* $\operatorname{Ker}\varphi_{(M,f)}$ *is the maximal subrepresentation of* M *contained in* $\operatorname{Ker} f$.

PROOF. First we have to show that $\operatorname{Ker}\varphi_{(M,f)}$ is a subrepresentation of M. Set $U_i := \operatorname{Ker}\varphi_i$ then

$$M_\alpha(U_{s_\alpha}) \subseteq U_{t_\alpha}$$

holds for all $\alpha \in Q_1$ since $\operatorname{Ker}\varphi_{(M,f)}$ is a morphism of Q representation and commutes with the maps M_α. It remains to show that U is maximal in $\operatorname{Ker} f$. Let $N \subseteq \operatorname{Ker} f$ a subrepresentation of M. Hence it satisfies

$$M_\alpha(N_{s_\alpha}) \subseteq N_{t_\alpha} \text{ for all } \alpha \in Q_1$$

and this is also true for concatenations of arrows, i.e.

$$M_p(N_{s_p}) \subseteq N_{t_p} \text{ for all } p \in P(Q,I).$$

Since N is contained in $\operatorname{Ker} f$ it also satisfies $f_i(N_i) = 0$ which implies

$$f_j \circ M_p(N_i) = 0 \text{ for all } p \in P_{i,j}(Q,I).$$

By the definition of $\varphi_{(M,f)}$ this yields $\varphi_i(N_i) = 0$ for all $i \in Q_0$ and hence we obtain $N \subseteq \operatorname{Ker}\varphi_{(M,f)}$. Accordingly the subrepresentation $\operatorname{Ker}\varphi_{(M,f)}$ is the maximal subrepresentation of M which is contained in $\operatorname{Ker} f$. $\square$

COROLLARY 2.9. The map $\varphi_{(M,f)} : M \to J$ is injective if and only if the pair (M,f) is stable.

PROOF. By definition of stability for the pair (M,f), the kernel $\operatorname{Ker} f$ contains no proper subrepresentation of M. The kernel $\operatorname{Ker}\varphi_{(M,f)} \subseteq M$ is a subrepresentation of M and maximal among the subrepresentations of M which are contained in $\operatorname{Ker} f \subseteq M$ as shown in the lemma above. Since the only subrepresentation of M contained in $\operatorname{Ker} f$ is the zero representation, the kernel of $\varphi_{(M,f)}$ is zero and the map is injective.

If $\varphi_{(M,f)} : M \to J$ is injective, the kernel of $\varphi_{(M,f)}$ is zero. By the above lemma, this is the maximal subrepresentation of M contained in $\operatorname{Ker} f$. Hence $\operatorname{Ker} f$ contains only the zero representation and the pair (M,f) is stable. $\square$

This implies that the image of $\mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I)$ under the map Φ lives inside the set $\operatorname{Hom}^0_Q(\mathbf{e},J)$ which contains the injective Q-morphisms to J. Now we have to show that the image contains all injective Q-morphisms.

PROPOSITION 2.10. The restricted map

$$\Phi : \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I) \to \operatorname{Hom}^0_Q(\mathbf{e},J)$$

is a bijection.

PROOF. The set of paths $P_{i,i}(Q,I)$ contains the empty path ϵ_i because the ideal I is admissible and can not contain empty paths. By convention the map M_{ϵ_i} is equal to the identity map. We can split the maps φ_i into components indexed by $j \in Q_0$ and $p \in P_{i,j}(Q,I)$. The component of φ_i indexed by i and ϵ_i equals the map f_i. Hence we can recover the maps f_i form φ_i by applying the projection to φ_i which only keeps the component indexed by i and ϵ_i. We want to denote this projection by $\mathrm{pr}_{\epsilon_i} : \Bbbk^{n_i} \to \Bbbk^{d_i}$.

The first component of the map Φ is the identity map on the variety of bound quiver representations. Accordingly two pairs (M,f) and (N,g) can not have the same image under Φ if $M \neq N$. By the study of the second component of Φ as done above, it is clear that $\varphi_{(M,f)} = \varphi_{(N,g)}$ is only possible if $f_i = g_i$ for all $i \in Q_0$. Otherwise the components of $\varphi^{(i)}_{(M,f)}$ and $\varphi^{(i)}_{(N,g)}$ which are indexed by i and ϵ_i can not be equal since they are given by f_i and g_i. This shows that the map Φ is injective.

Now let (U, ψ) be an element of $\mathrm{Hom}^0_Q(\mathbf{e}, J)$. We have to find a pair $(N, f) \in \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q, I)$ which is mapped to (U, ψ). Since Φ is acting as identity on the first component we can choose $N := U$ and define $f_i := \mathrm{pr}_{\epsilon_i} \circ \psi_i$. Then f_i is injective and $\mathrm{Ker}\, f \subseteq N$ only contains the zero subrepresentation since ψ_i is injective. Thus we have found a stable pair (N, f) satisfying that $\Phi(N, f) = (U, \psi)$ which proves that Φ is surjective. Here the equality $\varphi_{(N,f)} = \psi$ follows since both maps are morphisms of Q-representations and hence commute with maps N_α and J_α for all $\alpha \in Q_1$ and thus these commutativity relations also hold for paths. This allows to deduce the equality of the maps $\varphi_{(N,f)}$ and ψ from the equalities of the components indexed by i and ϵ_i which are equal by definition together with the structure of the map Φ. □

Proposition 2.11. The restricted map

$$\Phi : \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q, I) \to \mathrm{Hom}^0_Q(\mathbf{e}, J)$$

is $\mathrm{GL}_{\mathbf{e}}$-equivariant.

Proof. The action of the group $\mathrm{GL}_{\mathbf{e}}$ on $\mathrm{R}_{\mathbf{e}}(Q, I)$ extends to the action

$$\begin{aligned} \mathrm{GL}_{\mathbf{e}} \times \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q, I) &\to \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q, I) \\ \big(g, (M, f)\big) &\mapsto (g.M, g.f) \end{aligned}$$

where

$$g.M := \big(g_{t_\alpha} M_\alpha g^{-1}_{s_\alpha}\big)_{\alpha \in Q_1} \quad \text{and} \quad g.f := (f_i g_i^{-1})_{i \in Q_0}.$$

On the variety of injective Q-morphisms to J the group $\mathrm{GL}_{\mathbf{e}}$ acts via

$$\begin{aligned} \mathrm{GL}_{\mathbf{e}} \times \mathrm{Hom}^0_Q(\mathbf{e}, J) &\to \mathrm{Hom}^0_Q(\mathbf{e}, J) \\ \big(g, (N, \psi)\big) &\mapsto (g.N, g.\psi) \end{aligned}$$

where

$$g.N := \big(g_{t_\alpha} N_\alpha g^{-1}_{s_\alpha}\big)_{\alpha \in Q_1} \quad \text{and} \quad g.\psi := (\psi_i g_i^{-1})_{i \in Q_0}.$$

To prove the $\mathrm{GL}_{\mathbf{e}}$-equivariance of Φ we have to show

$$g.\Phi(M, f) = \Phi\big(g.(M, f)\big).$$

The first component of $\Phi(M, f)$ is given by M and $\mathrm{GL}_{\mathbf{e}}$ acts on the first component of $\mathrm{Hom}^0_Q(\mathbf{e}, J)$ and $\mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q, I)$ in the same way. Hence the first component of the map Φ is $\mathrm{GL}_{\mathbf{e}}$-equivariant. For the second component we have to check

$$g.\varphi_{(M,f)} = \varphi_{g.(M,f)}.$$

By the definition of Φ and the $\mathrm{GL}_{\mathbf{e}}$-action this translates to the subsequent equations for all $i \in Q_0$

$$\begin{aligned}
\varphi^{(i)}_{(M,f)} g_i^{-1} &= \varphi^{(i)}_{(g.M,g.f)} \\
\Big(\bigoplus_{j\in Q_0}\bigoplus_{p\in P_{i,j}(Q,I)} f_j \circ M_p\Big) g_i^{-1} &= \bigoplus_{j\in Q_0}\bigoplus_{p\in P_{i,j}(Q,I)} f_j g_j^{-1} \circ (g.M)_p \\
\Big(\bigoplus_{j\in Q_0}\bigoplus_{p\in P_{i,j}(Q,I)} f_j \circ M_p\Big) g_i^{-1} &= \bigoplus_{j\in Q_0}\bigoplus_{p\in P_{i,j}(Q,I)} f_j g_j^{-1} \circ g_j M_p g_i^{-1} \\
&= \bigoplus_{j\in Q_0}\bigoplus_{p\in P_{i,j}(Q,I)} f_j \circ M_p g_i^{-1}.
\end{aligned}$$

Here we got from the second line on the right hand side to the third line since a path $p \in P_{i,j}(Q,I)$ can be written as

$$p = \alpha_{j_k} \circ \cdots \circ \alpha_{j_2} \circ \alpha_{j_1}$$

where $i = s_p = s_{\alpha_{j_1}}$ and $j = t_p = t_{\alpha_{j_k}}$ for some integer k. This means that M_p is given as

$$M_p = M_{\alpha_{j_k}} \circ \cdots \circ M_{\alpha_{j_2}} \circ M_{\alpha_{j_1}}.$$

Hence for $(g.M)_p$ we obtain

$$\begin{aligned}
(g.M)_p &= g_{i_{k+1}} M_{\alpha_{j_k}} g_{i_k}^{-1} \circ g_{i_k} M_{\alpha_{j_{k-1}}} g_{i_{k-1}}^{-1} \circ \cdots \circ g_{i_3} M_{\alpha_{j_2}} g_{i_2}^{-1} \circ g_{i_2} M_{\alpha_{j_1}} g_{i_1}^{-1} \\
&= g_{i_{k+1}} M_{\alpha_{j_k}} \circ \cdots \circ M_{\alpha_{j_2}} \circ M_{\alpha_{j_1}} g_{i_1}^{-1} \\
&= g_j M_p g_i^{-1}
\end{aligned}$$

where we write $i_q := s_{\alpha_{j_q}}$ and $i_{k+1} := t_{\alpha_{j_k}}$ for simplicity and used $j = i_{k+1}$ and $i_1 = i$ in the last step. □

Now we have collected all properties required to prove the isomorphism.

Proof of Theorem 2.2. We have a $\mathrm{GL}_{\mathbf{e}}$-equivariant isomorphism

$$\Phi : \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I) \to \mathrm{Hom}^0_Q(\mathbf{e},J)$$

which hence descends to an isomorphism of the geometric $\mathrm{GL}_{\mathbf{e}}$-quotients

$$\mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q,I) := \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I)/\mathrm{GL}_{\mathbf{e}} \cong \mathrm{Hom}^0_Q(\mathbf{e},J)/\mathrm{GL}_{\mathbf{e}} \cong \mathrm{Gr}^Q_{\mathbf{e}}(J).$$

Here the existence of the geometric quotient of the left hand side is shown by H. Nakajima in [**60**]. □

2.2. One Point Extensions and Deframing

In this section, we prove that the quotient map from the variety of extended quiver representations to the framed module space is smooth. This is required for the proof of Theorem 2.3. For this purpose we rewrite the framed moduli space in terms of ordinary quiver moduli. Therefore we have to add one additional point and certain additional arrows to the original quiver. This technique was called **deframing** by W. Crawley-Boevey in [**24**]. In this section we follow the approach of M. Reineke as presented in the articles [**62, 63**].

As before, let Q be a finite quiver and $I \subset \Bbbk Q$ an admissible ideal of bounding relations. A linear function

$$\Theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

is called **stability** for Q. The dimension of Q-representation is given by the function

$$\begin{aligned} \dim : \operatorname{rep}_{\Bbbk}(Q,I) &\longrightarrow \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z} \\ U &\mapsto \mathbf{d} := \mathbf{dim}\, U \mapsto \sum_{i\in Q_0} d_i. \end{aligned}$$

Combining both functions the **slope** σ_Θ of a Q-representation is obtained by the function

$$\begin{aligned} \sigma_\Theta : \operatorname{rep}_{\Bbbk}(Q,I) &\longrightarrow \mathbb{Q} \\ U &\longmapsto \frac{\Theta(\mathbf{dim}\, U)}{\dim U}. \end{aligned}$$

If we are working with a fixed stability Θ we drop the index and just write σ.

DEFINITION 2.12. A Q-representation $V \in \operatorname{rep}_{\Bbbk}(Q,I)$ is called

a) σ-**stable** if $\sigma(U) < \sigma(V)$ for all proper subrepresentations U of V and
b) σ-**semistable** if $\sigma(U) \le \sigma(V)$ for all proper subrepresentations U of V.

For a dimension vector $\mathbf{d}$, the variety of σ-stable Q-representations is given by

$$\mathrm{R}^{\sigma-s}_{\mathbf{d}}(Q,I) := \big\{V \in \mathrm{R}_{\mathbf{d}}(Q,I) : V \text{ is stable}\big\}$$

and the variety of σ-semistable Q-representations is given by

$$\mathrm{R}^{\sigma-ss}_{\mathbf{d}}(Q,I) := \big\{V \in \mathrm{R}_{\mathbf{d}}(Q,I) : V \text{ is semistable}\big\}.$$

Some properties of the variety of σ-(semi-)stable Q-representations are collected in the subsequent theorem. The proofs can be found in the article by A. King [**49**] and this formulation is due to M. Reineke [**63**, Theorem 3.2]. In the article by M. Reineke the statement is restricted to the case of finite quivers without oriented cycles. The original article by A. King is written in the generality of finite dimensional algebras, i.e. finite quivers with an admissible set of relations. This is the generality we need for our application.

THEOREM 2.13. *The σ-semistable locus $\mathrm{R}^{\sigma-ss}_{\mathbf{d}}(Q,I)$ is an open subset of the variety of bound quiver representations $\mathrm{R}_{\mathbf{d}}(Q,I)$ and the σ-stable locus $\mathrm{R}^{\sigma-s}_{\mathbf{d}}(Q,I)$ is an open subset of $\mathrm{R}^{\sigma-ss}_{\mathbf{d}}(Q,I)$. There exists an algebraic quotient $\mathrm{M}^{\sigma-ss}_{\mathbf{d}}(Q,I)$ of $\mathrm{R}^{\sigma-ss}_{\mathbf{d}}(Q,I)$ by the group $\mathrm{GL}_{\mathbf{d}}$ and a geometric quotient $\mathrm{M}^{\sigma-s}_{\mathbf{d}}(Q,I)$ of $\mathrm{R}^{\sigma-s}_{\mathbf{d}}(Q,I)$ by the group $\mathrm{GL}_{\mathbf{d}}$. The variety $\mathrm{M}^{\sigma-s}_{\mathbf{d}}(Q,I)$ embeds as an open subset into the projective variety $\mathrm{M}^{\sigma-ss}_{\mathbf{d}}(Q,I)$.*

In the version of this theorem by M. Reineke there is also a dimension formula for the variety $\mathrm{M}^{\sigma-ss}_{\mathbf{d}}(Q,I)$ and the quotient $\mathrm{M}^{\sigma-s}_{\mathbf{d}}(Q,I)$ is a smooth variety in his setting. These two statements are not true if we allow oriented cycles in our quiver Q. In this setting the path algebra $\Bbbk Q$ is not finite any more and we have to introduce bounding relations I in order to apply A. Kings theory for modules over finite dimensional algebras.

DEFINITION 2.14. For a finite quiver Q and a dimension vector $\mathbf{d} \in \mathbb{Z}^{Q_0}$ define the **one point extension** $\tilde{Q}(\mathbf{d})$. The set of vertices is given by adding one extra vertex

$$\tilde{Q}(\mathbf{d})_0 := Q_0 \cup \{\infty\}$$

and from every point i of Q we add d_i many framing arrows to the extra vertex, i.e.

$$\tilde{Q}(\mathbf{d})_1 := Q_1 \cup \Big\{\alpha_{i,k} : s_{\alpha_{i,k}} = i \text{ and } t_{\alpha_{i,k}} = \infty \text{ for all } k \in [d_i]\Big\}.$$

We denote the set of extending arrows which are added to the arrows of the original quiver by $E(Q,\mathbf{d})$. For a dimension vector $\mathbf{e} \in \mathbb{Z}^{Q_0}$ we define its extension $\tilde{\mathbf{e}} \in \mathbb{Z}^{\tilde{Q}(\mathbf{d})_0}$ by

$$\tilde{e}_i := e_i \text{ for all } i \in Q_0 \text{ and } \tilde{e}_\infty := 1.$$

These one point extensions can be used to identify the framed quiver representations with classical representations of an extended quiver. Let I be an admissible ideal for the path algebra $\Bbbk Q$. Then I is also admissible for the path algebra $\Bbbk\tilde{Q}(\mathbf{d})$ of the extended quiver $\tilde{Q}(\mathbf{d})$.

Proposition 2.15. There is an isomorphism between the variety of representations of the extended quiver and the variety of extended quiver representations, i.e.

$$\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}), I\big) \cong \mathrm{R}_{\mathbf{e},\mathbf{d}}(Q, I).$$

Proof. The variety of quiver representations for the extended quiver is defined as

$$\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d})\big) := \bigoplus_{\alpha \in \tilde{Q}(\mathbf{d})_1} \mathrm{Hom}_\Bbbk\left(\Bbbk^{\tilde{e}_{s\alpha}}, \Bbbk^{\tilde{e}_{t\alpha}}\right).$$

The set of arrows of the extended quiver consists of the arrows of the original quiver and the extending arrows. So we can rewrite the variety of quiver representations as

$$\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d})\big) = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_\Bbbk\left(\Bbbk^{\tilde{e}_{s\alpha}}, \Bbbk^{\tilde{e}_{t\alpha}}\right) \oplus \bigoplus_{\alpha \in E(Q,\mathbf{d})} \mathrm{Hom}_\Bbbk\left(\Bbbk^{\tilde{e}_{s\alpha}}, \Bbbk^{\tilde{e}_{t\alpha}}\right)$$

The dimension vector of the extended quiver is defined as

$$\tilde{e}_i := e_i \text{ for all } i \in Q_0 \text{ and } \tilde{e}_\infty := 1$$

and the extending arrows all head towards the vertex ∞ which yields

$$\begin{aligned}
\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d})\big) &= \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_\Bbbk\left(\Bbbk^{e_{s\alpha}}, \Bbbk^{e_{t\alpha}}\right) \oplus \bigoplus_{\alpha \in E(Q,\mathbf{d})} \mathrm{Hom}_\Bbbk\left(\Bbbk^{e_{s\alpha}}, \Bbbk\right) \\
&= \mathrm{R}_{\mathbf{e}}(Q) \oplus \bigoplus_{i \in Q_0} \bigoplus_{k \in [d_i]} \mathrm{Hom}_\Bbbk\left(\Bbbk^{e_i}, \Bbbk\right) \\
&\cong \mathrm{R}_{\mathbf{e}}(Q) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}_\Bbbk\left(\Bbbk^{e_i}, \Bbbk^{d_i}\right) \\
&= \mathrm{R}_{\mathbf{e},\mathbf{d}}(Q).
\end{aligned}$$

The relations in the ideal I only effect the first part of the direct sum decomposition of the variety of quiver representations of the extended quiver such that every step also works for the varieties of bounded quiver representations. □

In order to carry this isomorphism to the geometric quotients we have to identify the group actions on these varieties.

Proposition 2.16. On the variety of extended quiver representations

$$\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}), I\big)$$

the orbits of the groups $\mathrm{PGL}_{\tilde{\mathbf{e}}} := \mathrm{GL}_{\tilde{\mathbf{e}}}/\Bbbk^*$ and $\mathrm{GL}_{\mathbf{e}}$ coincide.

PROOF. The group $\mathrm{GL}_{\tilde{\mathbf{e}}}$ acts on the variety of representations of the extended quiver via

$$\begin{aligned}\mathrm{GL}_{\tilde{\mathbf{e}}} \times \mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}), I\big) &\to \mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}), I\big)\\ \big(g, M\big) &\mapsto g.M\end{aligned}$$

where

$$g.M := \big(g_{t_\alpha} M_\alpha g_{s_\alpha}^{-1}\big)_{\alpha\in\tilde{Q}(\mathbf{d})_1}.$$

The action of the group $\mathrm{GL}_{\mathbf{e}}$ on the variety of extended quiver representations was defined as

$$\begin{aligned}\mathrm{GL}_{\mathbf{e}} \times \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q, I) &\to \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q, I)\\ \big(g, (N, f)\big) &\mapsto (g.N, g.f)\end{aligned}$$

where

$$g.N := \big(g_{t_\alpha} N_\alpha g_{s_\alpha}^{-1}\big)_{\alpha\in Q_1} \quad \text{and} \quad g.f := (f_i g_i^{-1})_{i\in Q_0}.$$

By the isomorphism between both varieties, each M in the variety of representations of the extended quiver corresponds to a pair (N, f) in the variety of extended quiver representations. This isomorphism is given by the subsequent identifications

$$M_\alpha = N_\alpha \text{ for } \alpha \in Q_0 \text{ and } M_\alpha = f_i \text{ if } s_\alpha = i \text{ and } t_\alpha = \infty.$$

This induces an action of the group $\mathrm{GL}_{\mathbf{e}}$ on the variety

$$\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}), I\big)$$

where $g \in \mathrm{GL}_{\mathbf{e}}$ acts via

$$g.M := \begin{cases} M_\alpha g_{s_\alpha}^{-1} & \text{if } t_\alpha = \infty\\ g_{t_\alpha} M_\alpha g_{s_\alpha}^{-1} & \text{otherwise.}\end{cases}$$

The above isomorphism is $\mathrm{GL}_{\mathbf{e}}$-equivariant with respect to this action.

Every element of the group $\mathrm{GL}_{\tilde{\mathbf{e}}}$ can be written as a pair (g, λ) where $g \in \mathrm{GL}_{\mathbf{e}}$ and $\lambda \in \Bbbk^*$ because

$$\mathrm{GL}_{\tilde{\mathbf{e}}} := \prod_{i\in\tilde{Q}(\mathbf{d})_0} \mathrm{GL}_{\tilde{e}_i}(\Bbbk) = \prod_{i\in Q_0} \mathrm{GL}_{e_i}(\Bbbk) \times \mathrm{GL}_{e_\infty}(\Bbbk) = \mathrm{GL}_{\mathbf{e}} \times \mathrm{GL}_1(\Bbbk) = \mathrm{GL}_{\mathbf{e}} \times \Bbbk^*.$$

The group $\mathrm{PGL}_{\tilde{\mathbf{e}}}$ is obtained by the relation

$$(g, \lambda) \sim (h, \nu) :\Leftrightarrow \text{ There exists a } \mu \in \Bbbk^* \text{ s.t. } \big((\mu g_i)_{i\in Q_0}, \mu\lambda\big) = \big((h_i)_{i\in Q_0}, \nu\big)$$

on the elements of the group $\mathrm{GL}_{\tilde{\mathbf{e}}}$. With this definition of the group $\mathrm{PGL}_{\tilde{\mathbf{e}}}$, its action on the variety

$$\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}), I\big)$$

is independent of the choice of the representative, i.e.

$$\begin{aligned}(\mu g).M &= \big((\mu g_{t_\alpha}) M_\alpha (\mu g_{s_\alpha})^{-1}\big)_{\alpha\in\tilde{Q}(\mathbf{d})_1} = \big(\mu g_{t_\alpha} M_\alpha g_{s_\alpha}^{-1} \mu^{-1}\big)_{\alpha\in\tilde{Q}(\mathbf{d})_1}\\ &= \big(\mu\mu^{-1} g_{t_\alpha} M_\alpha g_{s_\alpha}^{-1}\big)_{\alpha\in\tilde{Q}(\mathbf{d})_1} = \big(g_{t_\alpha} M_\alpha g_{s_\alpha}^{-1}\big)_{\alpha\in\tilde{Q}(\mathbf{d})_1}\\ &= g.M.\end{aligned}$$

Hence in every class we can take $(h, 1)$ with $h \in \mathrm{GL}_{\mathbf{e}}$ as representative and the action of $(h, 1)$ coincides with the action of h as element of the group $\mathrm{GL}_{\mathbf{e}}$ with its action on the variety of representations of the extended quiver as introduced above. □

COROLLARY 2.17. The isomorphism

$$\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}), I\big) \cong \mathrm{R}_{\mathbf{e},\mathbf{d}}(Q, I).$$

is $\mathrm{GL}_{\mathbf{e}}$-equivariant and the $\mathrm{GL}_{\tilde{\mathbf{e}}}$-orbits and $\mathrm{PGL}_{\tilde{\mathbf{e}}}$-orbits in the variety of representations of the extended quiver coincide.

With the same methods as in the proof of the above proposition it is checked that the additional $\Bbbk^*$-action on the extending vertex has no effect on the orbits. Given a $\lambda \in \Bbbk^*$ acting on the vector space over the extending vertex rescaling the matrices acting on the spaces over the points of the original quiver by the same parameter leads to the same point in the orbit. Moreover the $\mathrm{GL}_{\mathbf{e}}$-equivariant isomorphism of these varieties is compatible with the notions of stability on both sides such that it descends to an isomorphism of the quotients.

PROPOSITION 2.18. For the stability

$$\Theta : \mathrm{rep}_{\Bbbk}\big(\tilde{Q}(\mathbf{d}), I\big) \to \mathbb{Z}$$
$$U \mapsto -(\dim U)_\infty.$$

there is an isomorphism

$$\mathrm{M}^{\sigma_\Theta - ss}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}), I\big) \cong \mathrm{M}^{s}_{\mathbf{e},\mathbf{d}}(Q, I).$$

PROOF. In the previous proposition we have show that the variety of quiver representations of the extended quiver and the variety of framed quiver representations of the original quiver are isomorphic. Hence we have to relate the notions of stability on both sides and show that the isomorphism descends to an isomorphism of the quotients.

Let $V \in \mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}), I\big)$ be a bounded representation of the extended quiver $\tilde{Q}(\mathbf{d})$. By Proposition 2.15 we can view V as a pair $(N, f) \in \mathrm{R}_{\mathbf{e},\mathbf{d}}(Q, I)$ where $N \in \mathrm{R}_{\mathbf{e}}(Q, I)$ and all $f_i : \Bbbk^{e_i} \to \Bbbk^{d_i}$ are linear maps. With the stability Θ defined in this proposition the representation V has slope

$$\sigma_\Theta(V) = -\frac{1}{\dim N + 1}$$

since $e_\infty = 1$. Let U be a non-zero proper subrepresentation of V. For the slope of U we distinguish two different cases.

First we compute the slope for subrepresentations U with $\dim U_\infty = 1$. In terms of framed representations U can be written as a pair (M, h) and the slope computes as

$$\sigma_\Theta(U) = -\frac{1}{\dim M + 1} < -\frac{1}{\dim N + 1} = \sigma_\Theta(V)$$

since U is a proper subrepresentation of V.

Now we consider subrepresentations U with $\dim U_\infty = 0$ and obtain their slope as

$$\sigma_\Theta(U) = -\frac{0}{\dim M + 1} = 0 > -\frac{1}{\dim N + 1} = \sigma_\Theta(V).$$

These computation shows that for this definition of stability the σ_Θ-stable locus and the σ_Θ-semistable locus coincide. A representation V can only be stable if there exist no proper subrepresentations U of V such that the dimension of the subrepresentations over the extending vertex is zero. Hence U is unstable if there is a $U \subset V$ such that $\dim U_\infty = 0$.

Again we write $U = (M, h)$ and $V = (N, f)$. Since $\dim U_\infty = 0$ the framing maps h_i have to send everything to zero. Let $\iota : M \to N$ be an embedding. Since U is a subrepresentation of V the maps f_i have to send the image $\iota(M) \subset N$ to zero. Hence $\iota(M)$ is contained in the kernels of the maps f_i and the pair $(N, f) \in \mathrm{R}_{\mathbf{e},\mathbf{d}}(Q, I)$ is not stable.

Now let M be a proper subrepresentation of N which is included in the kernel of the maps f_i. Then $U := (M, f|_M)$ is a subrepresentation of $V = (N, f)$ in $\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}), I\big)$. Since $f|_M = 0$ we have $\dim U_\infty = 0$ and the representation V is not σ_Θ-stable. Hence the notions of stability on the two isomorphic varieties are equivalent and we obtain

$$\mathrm{R}_{\tilde{\mathbf{e}}}^{\sigma_\Theta - s}\big(\tilde{Q}(\mathbf{d}), I\big) = \mathrm{R}_{\tilde{\mathbf{e}}}^{\sigma_\Theta - ss}\big(\tilde{Q}(\mathbf{d}), I\big) \cong \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q, I).$$

The insomorphism is $\mathrm{GL}_{\mathbf{e}}$-equivariant and the $\mathrm{GL}_{\tilde{\mathbf{e}}}$-orbits and $\mathrm{GL}_{\mathbf{e}}$-orbits in the variety $\mathrm{R}_{\tilde{\mathbf{e}}}^{\sigma_\Theta - s}\big(\tilde{Q}(\mathbf{d}), I\big)$ coincide. Hence the isomorphism of the varieties above descends to an isomorphism of the geometric quotients

$$\begin{aligned}\mathrm{M}_{\tilde{\mathbf{e}}}^{\sigma_\Theta - ss}\big(\tilde{Q}(\mathbf{d}), I\big) &= \mathrm{R}_{\tilde{\mathbf{e}}}^{\sigma_\Theta - ss}\big(\tilde{Q}(\mathbf{d}), I\big)/\mathrm{GL}_{\tilde{\mathbf{e}}} = \mathrm{R}_{\tilde{\mathbf{e}}}^{\sigma_\Theta - ss}\big(\tilde{Q}(\mathbf{d}), I\big)/\mathrm{GL}_{\mathbf{e}} \\ &\cong \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q, I)/\mathrm{GL}_{\mathbf{e}} = \mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q, I).\end{aligned}$$

□

2.3. Free Group Action and Smooth Quotient Map

In this section we prove that the action of the group $\mathrm{GL}_{\mathbf{e}}$ on the variety of extended quiver representations is free and that the quotient map to the framed moduli space is smooth. The smoothness of the quotient map is required in order to apply the theorem by K. Bongartz concerning the preservation of geometric properties. For our purpose it is sufficient to work in the setting of algebraic varieties over the field $\Bbbk$. The statement we use to prove the smoothness of the quotient map works for algebraic schemes and we recall it in the full generality of its original formulation. Let G be an algebraic group and X an algebraic scheme over $\Bbbk$ and

$$\rho : G \times X \to X; (g, x) \mapsto g.x$$

an action of G on X. The statement we want to apply needs the subsequent notion of freeness as introduced in the book by D. Mumford [**59**, Definition 0.8.iv)].

Definition 2.19. The action ρ is called **free** if

$$\begin{aligned}\Psi : G \times X &\to X \times X \\ (g, x) &\mapsto (g.x, x)\end{aligned}$$

is a closed immersion.

This is more restrictive than the notion of set theoretical freeness where the map Ψ is only required to be injective.

Definition 2.20. Let (Y, π) be a geometric quotient of X by G. Assume that G is flat and of finite type over $S = \mathrm{Spec}(\Bbbk)$. If the quotient and the group action satisfy the properties:

(1) π is a flat morphism of finite type and

(2) Ψ is an isomorphism of $G \times X$ and $X \times_Y X$

we call X a **principal fibre bundle** over Y, with group G [**59**, Definition 0.10].

To show that the quotient map from the variety of framed quiver representations to the framed moduli space is smooth, we use the subsequent proposition by D. Mumford [**59**, Proposition 0.9].

PROPOSITION 2.21. Let G be an algebraic group and X be an algebraic variety over $\Bbbk$ If the action ρ of G on X is free and (Y, π) is a geometric quotient of X by G, then X is a principal fibre bundle over Y with group G.

If we can show that the action of $\mathrm{GL}_{\mathbf{e}}$ on $\mathrm{R}^{s}_{\mathbf{e},\mathbf{d}}(Q,I)$ is free in the sense of D. Mumford this proposition implies that the quotient map to the framed moduli space is a principal fibre bundle. This property is necessary to prove Theorem 2.3. The proof of the subsequent lemma follows the proof in the article by M. Reineke [**62**, Lemma 6.5].

LEMMA 2.22. *The action of* $\mathrm{PGL}_{\mathbf{e}}$ *on* $\mathrm{R}^{\sigma-ss}_{\mathbf{e}}(Q,I)$ *is free.*

PROOF. For the stability σ as introduced in the previous section we have seen that the stable and semistable locus coincide. Thus every representation $M \in \mathrm{R}^{\sigma-ss}_{\mathbf{e}}\big(Q,I\big)$ is already stable and the stabiliser of the group $\mathrm{GL}_{\mathbf{e}}$ acting on the representation M is isomorphic to $\Bbbk^*$. Accordingly the stabiliser is trivial for the action of the group $\mathrm{PGL}_{\mathbf{e}} := \mathrm{GL}_{\mathbf{e}}/\Bbbk^*$ and the map

$$\begin{aligned} \Psi : \mathrm{PGL}_{\mathbf{e}} \times \mathrm{R}^{\sigma-ss}_{\mathbf{e}}\big(Q,I\big) &\longrightarrow \mathrm{R}^{\sigma-ss}_{\mathbf{e}}\big(Q,I\big) \times \mathrm{R}^{\sigma-ss}_{\mathbf{e}}\big(Q,I\big) \\ (g,M) &\longmapsto (g.M, M) \end{aligned}$$

is injective. This proves that the action is set theoretically free.

It remains to show that the image of Ψ is closed and that Ψ and Ψ^{-1} are continuous. In the first component of the map Ψ we have a tuple of matrix multiplications and the second component is an identity. The inverse of a matrix and the multiplication of matrices admit polynomial descriptions for the entries of the resulting matrices. Hence both components of Ψ are continuous. Now we show that $\operatorname{Im}\Psi$ is closed. For the rest of the proof we use the abbreviations

$$\mathrm{R}^{ss}_{\mathbf{e}} := \mathrm{R}^{\sigma-ss}_{\mathbf{e}}(Q,I),\ \mathrm{E}_{\mathbf{e}} := \bigoplus_{i\in Q_0} \mathrm{End}_{\Bbbk}(\Bbbk^{e_i}),\ \mathrm{G} := \mathrm{GL}_{\mathbf{e}},\ \mathrm{PG} := \mathrm{PGL}_{\mathbf{e}}.$$

Define the map

$$\begin{aligned} \Phi : \mathrm{R}^{ss}_{\mathbf{e}} \times \mathrm{R}^{ss}_{\mathbf{e}} &\longrightarrow \mathrm{Hom}_{\Bbbk}(\mathrm{E}_{\mathbf{e}}, \mathrm{R}_{\mathbf{e}}) \\ (X,Y) &\longmapsto \Phi(X,Y) : \mathrm{E}_{\mathbf{e}} \longrightarrow \mathrm{R}_{\mathbf{e}} \\ &\qquad \phi = (\phi)_{i\in Q_0} \longmapsto \Phi(X,Y)\big((\phi)_{i\in Q_0}\big) := \big(\phi_{t_\alpha} X_\alpha - Y_\alpha \phi_{s_\alpha}\big)_{\alpha\in Q_1}. \end{aligned}$$

Any Q-morphism $\psi : X \to Y$ commutes with the maps X_α and Y_α by definition and hence is send to zero by the map Φ. This implies that the kernel of the map Φ is given by $\mathrm{Hom}_Q(X,Y)$ which is the space of all Q-morphisms from X to Y. The image of Ψ is the set

$$\operatorname{Im}\Psi := \big\{(g.M, M) : M \in \mathrm{R}^{ss}_{\mathbf{e}}, g \in \mathrm{PG}\big\}.$$

For some representative of $g \in \mathrm{PG}$ the corresponding tuple of matrices parametrises a Q-morphism $\phi^{(g)} : M \to g.M$ such that $\Phi(M, g.M)$ is non-trivial. We want to show that the image of Ψ has an equivalent characterisation based on the map Φ, i.e.

$$\operatorname{Im}\Psi = \big\{(Y,X) \in \mathrm{R}^{ss}_{\mathbf{e}} \times \mathrm{R}^{ss}_{\mathbf{e}} : \ker\Phi(X,Y) \neq \{0\}\big\} =: B.$$

But first we show that $\operatorname{Im}\Psi$ it is closed if it is equal to B. Here $\ker\Phi(X,Y)\neq\{0\}$ is equivalent to the condition that $\operatorname{rank}\Phi(X,Y)\leq m-1$ where m is the maximal rank for the elements of $\operatorname{Hom}_{\Bbbk}(\mathrm{E}_{\mathbf{e}},\mathrm{R}_{\mathbf{e}})$. Accordingly the image of Ψ is described as

$$\operatorname{Im}\Psi=\Phi^{-1}\Big(\big\{A\in\operatorname{Hom}_{\Bbbk}(\mathrm{E}_{\mathbf{e}},\mathrm{R}_{\mathbf{e}}):\operatorname{rank}A\leq m-1\big\}\Big).$$

The set on the right hand side is closed in $\operatorname{Hom}_{\Bbbk}(\mathrm{E}_{\mathbf{e}},\mathrm{R}_{\mathbf{e}})$ and hence its preimage in $\mathrm{R}^{ss}_{\mathbf{e}}\times\mathrm{R}^{ss}_{\mathbf{e}}$ is closed because the map Φ is continuous as its components have polynomial descriptions. This proves that the image of Ψ is closed under the assumption of the above parametrisation.

It remains to show that the image of Ψ is equal to B and that there exists a continuous inverse to Ψ. The set B is covered by open subsets $U_{I,J}$ for $I,J\in\binom{[m]}{r}$ which contain the pairs (X,Y) where the $I\times J$ minor of $\Phi(X,Y)$ is non-vanishing. For a fixed minor we can construct a local inverse of the map $\Phi(X,Y)$ and can recover a tuple non-zero matrices in $\mathrm{E}_{\mathbf{e}}$ satisfying the commutativity relations of $\operatorname{Hom}_Q(X,Y)$. From this tuple of matrices we can construct the unique element of the group PG which sends X to Y. In this way we can locally invert the morphism Ψ. Analogous to the inverse of a matrix this local inverse Ψ^{-1} has a polynomial description and hence is continuous. □

Based on the technique of deframing we can use this lemma to prove the freeness of the group action on the variety of extended quiver representations.

Corollary 2.23. The action of $\mathrm{GL}_{\mathbf{e}}$ on $\mathrm{R}^{s}_{\mathbf{e},\mathbf{d}}(Q,I)$ is free.

Proof. In Proposition 2.15 we have identified the action of $\mathrm{GL}_{\mathbf{e}}$ on $\mathrm{R}_{\mathbf{e}}(Q,I)$ with the action of $\mathrm{PGL}_{\tilde{\mathbf{e}}}$ on $\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}),I\big)$. For the stability σ_Θ as introduced above we have the isomorphism

$$\mathrm{R}^{s}_{\mathbf{e},\mathbf{d}}(Q,I)\cong\mathrm{R}^{\sigma_\Theta-ss}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}),I\big)$$

and to the $\mathrm{PGL}_{\tilde{\mathbf{e}}}$-action on the latter variety we can apply Lemma 2.22. □

Finally we can put everything together.

Corollary 2.24. The set $\mathrm{R}^{s}_{\mathbf{e},\mathbf{d}}(Q,I)\subseteq\mathrm{R}_{\mathbf{e},\mathbf{d}}(Q,I)$ is open. The quotient map

$$\begin{aligned}\pi:\mathrm{R}^{s}_{\mathbf{e},\mathbf{d}}(Q,I)&\longrightarrow\mathrm{M}^{s}_{\mathbf{e},\mathbf{d}}(Q,I)\cong\mathrm{Gr}^{Q}_{\mathbf{e}}(J)\\(N,f)&\longmapsto[N,f]:=\mathrm{GL}_{\mathbf{e}}.(N,f)\end{aligned}$$

is a principal $\mathrm{GL}_{\mathbf{e}}$-bundle. In particular, π is a smooth morphism.

Proof. We have identified $\mathrm{R}^{s}_{\mathbf{e},\mathbf{d}}(Q,I)$ with $\mathrm{R}^{\sigma-ss}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}),I\big)$ which is open inside $\mathrm{R}_{\tilde{\mathbf{e}}}\big(\tilde{Q}(\mathbf{d}),I\big)$ by Theorem 2.13. The latter space is identified with $\mathrm{R}_{\mathbf{e},\mathbf{d}}(Q,I)$ and hence the variety of stable framed representations is open in this space. By Corollary 2.23 we know that $\mathrm{GL}_{\mathbf{e}}$ acts freely on $\mathrm{R}^{s}_{\mathbf{e},\mathbf{d}}(Q,I)$. Applying the proposition by D. Mumford we arrive at the desired statement. □

2.4. Non-emptiness of the Framed Moduli Space

The interpretation of the quiver Grassmannian as framed moduli space allows to lift geometric properties from the variety of quiver representations to the corresponding quiver Grassmannian. Let Q be a finite quiver and take a dimension vector $\mathbf{e}\in\mathbb{Z}^{Q_0}$ with at least one positive entry. Then the variety of quiver representations is non-empty since it always contains the direct sum of simple representations

$$S_{\mathbf{e}} := \bigoplus_{i \in Q_0} S_i \otimes \Bbbk^{e_i}$$

where the simple representation S_i has the vector space $\Bbbk$ over the i-th vertex and the rest is zero.

For quiver Grassmannians the question of non-emptiness is much harder to answer since it depends on the choice of dimension vector of the subrepresentations and the shape of the indecomposable embeddings. For equioriented quivers of type A these embeddings are easy to understand but in general their shape can be arbitrarily complicated. This makes it hard to give a general criterion for the non-emptiness of quiver Grassmannians. For Dynkin quivers there exists a criterion by K. Möllenhoff and M. Reineke [**58**, Theorem 1.1].

If the quiver Grassmannian admits a description as framed moduli space there is a criterion based on this parametrisation. In the setting of Dynkin quivers the subsequent Lemma is proven by M. Reineke in [**63**, Lemma 4.1]. With the same methods it is possible to prove the generalisation to modules over the bounded quiver algebra of a finite connected quiver.

Lemma 2.25. *Given a representation M, there exists a map $f : M \to V$ making the pair (M, f) stable if and only if*

$$\dim \operatorname{Hom}_Q \big(S_i, M\big) \le d_i \text{ for all } i \in Q_0.$$

For a pair (M, f) in the framed module space $\mathrm{R}_{\mathbf{e},\mathbf{d}}(Q, I)$ it is possible to define the map

$$\varphi_{(M,f)} : M \to J$$

as in Definition 2.7. This map is injective if and only if the pair (M, f) is stable as shown in Corollary 2.9. Hence we arrive at the following corollary.

Corollary 2.26. Given a representation M, there exists an embedding

$$\psi : M \hookrightarrow J$$

if and only if

$$\dim \operatorname{Hom}_Q \big(S_i, M\big) \le d_i \text{ for all } i \in Q_0.$$

This criterion for the existence of embeddings is important to parametrise the image of the projection from the quiver Grassmannian to the variety of quiver representations as done in Section 3.5 for the equioriented cycle. This parametrisation of the quiver Grassmannian is useful to determine strata of maximal dimension computationally. More details about these computations are given in the section where the parametrisation of the image is developed and in Section 3.2 where the framed moduli interpretation is applied to compute the dimension of certain quiver Grassmannians for the equioriented cycle.

2.5. Orbits and Strata of Quiver Representations

This section is devoted to the proof of Theorem 2.3 which links the action of the group $\mathrm{GL}_{\mathbf{e}}$ on the variety of quiver representations and the action of the automorphism group on the quiver Grassmannian. This link allows us to translate various geometric properties between both varieties.

The structure of the proof is based on the approach by M. Reineke [**63**]. All the steps in the proof are similar to the Dynkin case but have to be generalised to the setting of finite dimensional algebras, i.e. the setting of finite connected quivers Q with an admissible ideal I describing relations on the paths.

Let π be the map

$$\begin{aligned} \pi : \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I) &\longrightarrow \mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q,I) \cong \mathrm{Gr}^Q_{\mathbf{e}}(J) \\ (N,f) &\longmapsto [N,f] := \mathrm{GL}_{\mathbf{e}}.(N,f). \end{aligned}$$

Define $\mathrm{R}^{(\mathbf{d})}_{\mathbf{e}}(Q,I)$ as the image of the projection

$$\begin{aligned} pr : \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I) &\longrightarrow \mathrm{R}_{\mathbf{e}}(Q,I) \\ (N,f) &\longmapsto N. \end{aligned}$$

The automorphism group for a quiver representation M with dimension vector $\mathbf{e} := \mathbf{dim}\, M$ is defined as

$$\mathrm{Aut}_Q(M) := \big\{\psi \in \mathrm{Hom}_Q(M,M) : \psi_i : \Bbbk^{e_i} \to \Bbbk^{e_i} \text{ is bijective for all } i \in Q_0\big\}$$

and it is isomorphic to the stabiliser group of M in $\mathrm{GL}_{\mathbf{e}}$, i.e.

$$\mathrm{Aut}_Q(M) \cong \mathrm{Stab}_{\mathrm{GL}_{\mathbf{e}}}(M) := \big\{g \in \mathrm{GL}_{\mathbf{e}} : g.M = M\big\}.$$

LEMMA 2.27. *Given a representation M and two embeddings $\varphi, \psi \in \mathrm{Hom}^0_Q(M,J)$ of M into an injective representation J, any automorphism $a \in \mathrm{Aut}_Q(M)$ of M extends to an automorphism $A \in \mathrm{Aut}_Q(J)$ of J such that $A\varphi = \psi a$.*

PROOF. Let $J' \subseteq J$ be the injective hull of $\varphi(M) \subseteq J$, i.e. the smallest injective subrepresentation of J containing the image $\varphi(M)$. The quiver representation J is injective and can be written as direct sum of injective indecomposable representations of Q. Hence we can write it as $J = J' \oplus J''$ where J'' is the complement of J' in J. For the morphism φ there also exists a decomposition into the components φ' and φ'' such that $\varphi'(M) = \varphi(M)$ and $\varphi'' \equiv 0$.

By definition the injective hull of a representation is unique up to isomorphism. Hence the map

$$\psi \circ a : M \to J$$

factors into $(\psi \circ a)'$ and $(\psi \circ a)''$ with $(\psi \circ a)'(M) = \psi \circ a(M)$ and $(\psi \circ a)'' \equiv 0$ together with an injective hull $\tilde{J}'$ such that there exists an isomorphism $A' : J' \to \tilde{J}'$ satisfying $A'\varphi' = \psi a$.

Since J, J' and $\tilde{J}'$ are injective Q-representations which can be written as direct sums of indecomposable injective Q representations and J' and $\tilde{J}'$ are isomorphic there also exists an isomorphism

$$A'' : J'' \to \tilde{J}''.$$

The map A with components A' and A'' becomes an automorphism of J and satisfies $A\varphi = \psi a$ by construction. $\square$

LEMMA 2.28. *Two subrepresentations of an injective representation J are conjugated under the action of $\mathrm{Aut}_Q(J)$ if and only if they are isomorphic.*

PROOF. Following the universal Grassmannian construction, subrepresentations U of the representation J can be described by subspaces U_i in the vectorspaces J_i for $i \in Q_0$ which are compatible with the maps J_α for $\alpha \in Q_1$. The maps U_α can be defined as the restrictions of J_α, i.e. $U_\alpha := J_\alpha|_{U_{s_\alpha}}$.

An element A of the automorphism group $\mathrm{Aut}_Q(J)$ acts on J as

$$A.J = \Big(\big(A_i(J_i) \big)_{i\in Q_0}, \big(A_{t_\alpha} J_\alpha A_{s_\alpha}^{-1} \big)_{\alpha\in Q_0} \Big) = \Big(\big(J_i \big)_{i\in Q_0}, \big(J_\alpha \big)_{\alpha\in Q_0} \Big).$$

This induces an action on the subrepresentations of J by

$$A.U = \Big(\big(A_i(U_i) \big)_{i\in Q_0}, \big(J_\alpha|_{A_{s_\alpha}(U_{s_\alpha})} \big)_{\alpha\in Q_0} \Big).$$

Hence two subrepresentations U and V of J are conjugated by the automorphisms of J if and only if there exists an $A \in \mathrm{Aut}_Q(J)$ such that

$$A_i(U_i) = V_i$$

holds for all $i \in Q_0$ and the restrictions $a_i := A_i|_{U_i}$ which are invertible satisfy

$$a_{t_\alpha} U_\alpha a_{s_\alpha}^{-1} = a_{t_\alpha} J_\alpha|_{U_{s_\alpha}} a_{s_\alpha}^{-1} = J_\alpha|_{a_{s_\alpha}(U_{s_\alpha})} = J_\alpha|_{V_{s_\alpha}} = V_\alpha.$$

This proves that U and V are conjugated by some element of $\mathrm{GL}_{\mathbf{e}}$ where $\mathbf{e} := \mathbf{dim}\, U$ and hence they are isomorphic.

For the other direction let U and V be two isomorphic subrepresentations of J. Hence there exists an isomorphism $\rho : U \to V$ and two embeddings $\varphi : U \hookrightarrow J$ and $\phi : V \hookrightarrow J$ such that

$$\psi(U) := \phi \circ \rho(U) = \phi(V).$$

Now we can apply the previous lemma to U, φ and ψ and obtain an automorphism A of J conjugating the subrepresentations (U, φ) and (V, ϕ). □

The subsequent lemma is commonly known and for example applied by K. Bongartz to obtain Corollary 1 from Theorem 3 in [**8**].

LEMMA 2.29. *Let G and H be special algebraic groups and Z a $G \times H$-variety with two morphisms $\phi : Z \to X$ and $\psi : Z \to Y$ such that:*

(a) ϕ is a G-equivariant smooth H-quotient,

(b) ψ is a H-equivariant principal G-bundle,

(c) The image of a G-orbit in X under the map $\psi \circ \phi^{-1}$ is a H-orbit in Y. Then the map $\psi \circ \phi^{-1} : X \to Y$ induces a bijection between G-stable subvarieties of X and H-stable subvarieties of Y. This correspondence preserves and reflects closures, inclusions and types of singularities occurring in orbit closures.

In the setting of the article by K. Bongartz a stronger version of (a) is satisfied. Namely it is of the same form as part (b). But it is still sufficient to make the above assumption in order to obtain the properties of the orbit correspondence. It would even be sufficient to use analogous requirements for part (b) as in part (a) of the above version. We prove Theorem 2.3 by showing that the above conditions (a), (b), (c) are satisfied in our setting where $Z = \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q, I)$, $X = \mathrm{R}^{(\mathbf{d})}_{\mathbf{e}}(Q, I)$, $Y = \mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q, I)$, $G = \mathrm{GL}_{\mathbf{e}}$ and $H = \mathrm{Aut}_Q(J)$. For the original version it was sufficient to assume that the groups are algebraic and with the above version it still might be sufficient to assume that the algebraic groups are connected. But the groups in our application are even special such that it is not necessary to extend the generality of the above lemma.

PROOF OF THEOREM 2.3. The proof in the setting of bound quiver representations works in the same way as the proof for quivers without relations given by M. Reineke in [**63**]. Hence we prove the statement following his approach. We have

$$\begin{array}{ccccc} & \mathrm{GL}_{\mathbf{e}} \curvearrowright & \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I) & \curvearrowleft \mathrm{Aut}_Q(J) & \\ & \swarrow^{pr} & & \searrow^{\pi} & \\ \mathrm{GL}_{\mathbf{e}} \curvearrowright & \mathrm{R}^{(\mathbf{d})}_{\mathbf{e}}(Q,I) & & \mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q,I) & \curvearrowleft \mathrm{Aut}_Q(J) \end{array}$$

were π is the geometric $\mathrm{GL}_{\mathbf{e}}$ quotient and pr is the projection as defined above.

The projection pr factors into and open immersion

$$\iota : \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I) \hookrightarrow \mathrm{R}_{\mathbf{e},\mathbf{d}}(Q,I)$$

and a projection

$$p : \mathrm{R}_{\mathbf{e},\mathbf{d}}(Q,I) = \mathrm{R}_{\mathbf{e}}(Q,I) \times \mathrm{Hom}_{\Bbbk}(\mathbf{e},\mathbf{d}) \to \mathrm{R}_{\mathbf{e}}(Q,I)$$

along an affine space since the stability form the above definition describes an open subset of $\mathrm{R}_{\mathbf{e},\mathbf{d}}(Q,I)$ and the projection to the variety of quiver representations is given by forgetting the information about the framing which is encoded by an element in $\mathrm{Hom}_{\Bbbk}(\mathbf{e},\mathbf{d})$.

Open immersions are smooth and trivial fibrations where the fibre is a smooth variety are also smooth. Hence the map pr is smooth as the concatenation of two smooth morphisms. By the definition of the map pr it is clear that this map is $\mathrm{GL}_{\mathbf{e}}$-equivariant and it is $\mathrm{Aut}_Q(J)$-invariant since it forgets the framing in J. The surjectivity of pr follows from the definition of $\mathrm{R}^{(\mathbf{d})}_{\mathbf{e}}(Q,I)$ and we can apply the quotient criterion [**51**, Satz 3.4] to show that pr is a $\mathrm{Aut}_Q(J)$-quotient. It follows that all assumptions of (a) are satisfied.

The quotient map π is $\mathrm{Aut}_Q(J)$-equivariant because the action of $\mathrm{GL}_{\mathbf{e}}$ and the $\mathrm{Aut}_Q(J)$ action commute. This follows from the $\mathrm{GL}_{\mathbf{e}}$ equivariance of the isomorphism of $\mathrm{R}^s_{\mathbf{e},\mathbf{d}}(Q,I)$ and $\mathrm{Hom}^0_{Q,I}(\mathbf{e},J)$ and the properties of both actions on the second space. Moreover π is a principal $\mathrm{GL}_{\mathbf{e}}$-bundle and smooth by Corollary 2.24. Accordingly it satisfies part (b) of Lemma 2.29.

By Lemma 2.28 we know that two subrepresentations of J are conjugate under the action of $\mathrm{Aut}_Q(J)$ if and only if they are isomorphic. Hence the image of a $\mathrm{GL}_{\mathbf{e}}$-orbit in $\mathrm{R}^{(\mathbf{d})}_{\mathbf{e}}(Q,I)$ under the map

$$\pi \circ pr^{-1} : \mathrm{R}^{(\mathbf{d})}_{\mathbf{e}}(Q,I) \to \mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q,I)$$

is a $\mathrm{Aut}_Q(J)$-orbit in $\mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q,I)$. This proves that part (c) of Lemma 2.29 is satisfied. Moreover distinct $\mathrm{GL}_{\mathbf{e}}$-orbits cannot have the same $\mathrm{Aut}_Q(J)$-orbit as image. From the definition of $\mathrm{R}^{(\mathbf{d})}_{\mathbf{e}}(Q,I)$ it follows that all of the $\mathrm{Aut}_Q(J)$-orbits in $\mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q,I)$ are obtained in this way. Thus we have even shown the bijectivity of the map between $\mathrm{GL}_{\mathbf{e}}$-stable subvarieties of $\mathrm{R}^{(\mathbf{d})}_{\mathbf{e}}(Q,I)$ and $\mathrm{Aut}_Q(J)$-stable subvarieties of $\mathrm{M}^s_{\mathbf{e},\mathbf{d}}(Q,I)$. Hence all conditions of Lemma 2.29 are satisfied and we can apply it to obtain the statement of the theorem. ☐

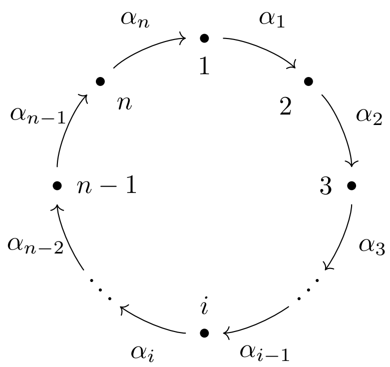
CHAPTER 3

The Equioriented Cycle

In this chapter we introduce nilpotent representations of the equioriented cycle and a class of quiver Grassmannians containing subrepresentations of certain nilpotent representations. These quiver Grassmannians are used to describe approximations of partial degenerations of the affine Grassmannian and the affine flag variety of type $\mathfrak{gl}_n$ in Chapter 5 and Chapter 6.

The geometric properties of these quiver Grassmannians are examined in Section 3.4. In Section 3.2 we develop a formula to compute the dimension of the space of morphisms between two nilpotent representations of the cycle based on word combinatorics for the representations. In Section 3.1 we recall results about the variety of quiver representations for nilpotent representations of the equioriented cycle which were obtained by G. Kempken in her thesis [**48**].

From now on let Δ_n be the quiver



The vertices of the equioriented cycle are in bijection with $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. For every $i \in \mathbb{Z}_n$ we define the path with ℓ arrows starting at vertex i as

$$p_i(\ell) := (i|\alpha_i\alpha_{i+1}\ldots\alpha_{i+\ell-1}|i+\ell).$$

The path algebra $\Bbbk\Delta_n$ is denoted by A_n. This algebra is not finite dimensional because there are paths $p_i(\ell)$ of arbitrary length around the cycle. To stay in the setting of finite dimensional algebras we have to define an admissible ideal of relations. Let

$$\mathrm{I}_N := \langle\, p_i(N) : i \in \mathbb{Z}_n \,\rangle \subset \Bbbk\Delta_n$$

be the ideal of the path algebra generated by all paths of length N. For $N \in \mathbb{N}$ we define the bounded path algebra $A_n^N := \Bbbk\Delta_n/\mathrm{I}_N$. The subsequent result is a special case of Theorem 1.2.

Proposition 3.1. The category $\mathrm{rep}_\Bbbk(\Delta_n, \mathrm{I}_N)$ of bounded quiver representations is equivalent to the category A_n^N-mod of (right) modules over the bounded path algebra.

Let $P_i \in \operatorname{rep}_{\Bbbk}(\Delta_n, \mathrm{I}_N)$ be the projective bounded representation of Δ_n at vertex $i \in \mathbb{Z}_n$. Define the projective representation

$$X := \bigoplus_{i \in \mathbb{Z}_n} P_i \otimes \Bbbk^{x_i}$$

where $x_i \in \mathbb{N}$ for all $i \in \mathbb{Z}_n$. Analogous let $I_j \in \operatorname{rep}_{\Bbbk}(\Delta_n, \mathrm{I}_N)$ be the injective bounded representation of Δ_n at vertex $j \in \mathbb{Z}_n$ and define the injective representation

$$Y := \bigoplus_{j \in \mathbb{Z}_n} I_j \otimes \Bbbk^{y_j}$$

with $y_j \in \mathbb{N}$ for all $j \in \mathbb{Z}_n$.

Throughout this chapter we study quiver Grassmannians of the form

$$\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$$

where $\mathbf{e} := \mathbf{dim}\, X$ is the dimension vector of X. This class of quiver Grassmannians is similar to the class of quiver Grassmannians for Dynkin quivers studied by G. Cerulli Irelli, E. Feigin and M. Reineke in [**20**]. The main difference is that here we take a non Dynkin quiver which additionally has an oriented cycle such that its path algebra is not finite any more.

It turns out that this generality is sufficient to provide finite dimensional approximations of the affine Grassmannian and the affine flag variety and their degenerations in type $\mathfrak{gl}_n$. This is done in Chapter 5 and Chapter 6. One key to the study of these varieties is the observation that all bounded projective representations of the equioriented cycle are bounded injective representations.

Let $M \in \operatorname{rep}_{\Bbbk}(\Delta_n)$ be a representation of the equioriented cycle. By definition it consists of a tuple of vector spaces $(V_i)_{i \in \mathbb{Z}_n}$ and a tuple of linear maps

$$\Big(M_i : V_i \to V_{i+1} \Big)_{i \in \mathbb{Z}_n}.$$

Define $V := \oplus_{i \in \mathbb{Z}_n} V_i$ and the linear map $A_M : V \to V$ which is of the form

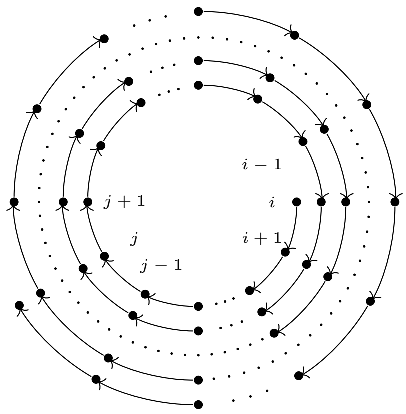
$$A_M := \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & M_n \\ M_1 & 0 & \dots & 0 & 0 & 0 \\ & M_2 & \ddots & \vdots & \vdots & \vdots \\ & & \ddots & 0 & 0 & 0 \\ & 0 & & M_{n-2} & 0 & 0 \\ & & & & M_{n-1} & 0 \end{pmatrix}$$

where all blocks in the matrix below the diagonal with the blocks M_i for $i \in [n-1]$ are equal to zero. The representation $M \in \operatorname{rep}_{\Bbbk}(\Delta_n)$ is called **nilpotent** if there exists an integer $\ell \in \mathbb{N}$ such that $A_M^{\ell} = 0$. This is equivalent to the condition that $M_{p_i(\ell)} = 0$ for all $i \in \mathbb{Z}_n$. Accordingly the nilpotent representations of the cycle are representations of the bound quiver (Δ_n, I_N) for some bounding parameter $N \in \mathbb{Z}$.

For $\ell \in \mathbb{N}$ and $i \in \mathbb{Z}_n$ define $V := \Bbbk^{\ell}$ and the map $A : V \to V$. It acts on the standard basis vectors of V by

$$A(e_j) := \begin{cases} e_{j+1} & \text{if } j < \ell \\ 0 & \text{if } j = \ell. \end{cases}$$

This describes a representation of Δ_n by the decomposition of V into the spaces V_i for $i \in \mathbb{Z}_n$ induced by $e_k \in V_{i+k-1}$. This representation of the oriented cycle is denoted by $U_i(\ell)$. For $j := i + \ell - 1$ in $\mathbb{Z}_n$ this decomposition of V and the map A are represented by the following picture where the dots correspond to the basis vectors e_j for $j \in [\ell]$ and an arrow indicates the map $e_j \to e_{j+1}$ and the separation of the dots into tuples over the vertices of the quiver represent the basis vectors for the spaces V_i.



Proposition 3.2 (Proposition 3.24 in [**67**]; Theorem 7.6 in [**50**]). All indecomposable nilpotent representations of Δ_n are of the form $U_i(\ell)$ for $i \in \mathbb{Z}_n$ and $\ell \in \mathbb{N}$.

These graphical interpretations of quiver representations are called coefficient quivers. Under certain circumstances they turn out to be useful in the computation of the Euler characteristics of quiver Grassmannians. Coefficient quivers are formally introduced in Definition 4.8. We give some examples and applications in Section 4.5. In the Chapter 5 and Chapter 6 they are used to compute the Poincaré polynomials for the approximations of the affine Grassmannian and the affine flag variety. Based on this description of the indecomposable nilpotent representations of the equioriented cycle we can prove that projective and injective representations are isomorphic. This is a direct consequence of their definition based on paths in the quiver as given in Section 1.4.

Corollary 3.3. For $n, N \in \mathbb{N}$ and all $i, j \in \mathbb{Z}_n$ the projective and injective representations P_i and I_j of the bound quiver (Δ_n, I_N) satisfy

$$P_i \cong U_i(N) \cong I_{i+N-1} \quad \text{and} \quad I_j \cong U_{j-N+1}(N) \cong P_{j-N+1}.$$

This allows us to apply Theorem 2.2 to the class of quiver Grassmannians as introduced above. Accordingly it is possible to realise these quiver Grassmannians as framed moduli spaces

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(X \oplus Y) \cong \mathrm{M}^s_{\mathbf{e},\mathbf{d}}(\Delta_n, \mathrm{I}_N)$$

where $d_i := y_i + x_{i-N+1}$. With this identification we are able to deduce results about the geometric properties of the quiver Grassmannian from the variety of quiver representations using Theorem 2.3. The latter variety was studied by G. Kempken in her thesis [**48**]. Below we recall some of her results about orbit closures and singularities.

3.1. Orbits and Singularities in the Variety of Quiver Representations

In this section we collect some of the main ideas from the thesis of G. Kempken [**48**]. Originally it was published in German and can not be found online and is only available in a few libraries. If not pointed out differently, everything which is written in this section can be found in her thesis and the translation is as close to the original text as possible. The aim of her work is to describe the orbit structure of $\mathrm{R}_{\mathbf{e}}(\Delta_n)$ and the types of singularities occurring in the orbit closures. We want to use her results to derive informations about the structure of the quiver Grassmannians for Δ_n as introduced in above. The remarks in this section are added to the original material to point out the connection to quiver Grassmannians.

Let Z be an irreducible variety of dimension n and $f : W \to Z$ is a resolution of singularities, i.e. W is smooth and f is proper and birational. Z has **rational singularities** if Z is normal and the higher direct images $R^i f_* \mathcal{O}_W$ of the structure sheaf $\mathcal{O}_W$ vanish for $i > 0$.

THEOREM 3.4 ([**46**], p. 50). *Z has rational singularities if and only if*

a) *Z is normal and Cohen-Macaulay.*

b) *For every n-form w defined on the smooth points of Z it is possible to extend f^*w to W.*

THEOREM 3.5 (p. 67). *The orbit closures inside* $\mathrm{R}_{\mathbf{e}}(\Delta_n)$ *have rational singularities.*

REMARK. For a representation $U \in \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ the closure of the orbit inside $\mathrm{R}_{\mathbf{e}}(\Delta_n)$ and its closure in $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ coincide such that we can apply this theorem to bounded quiver representations. By application of Theorem 2.3 by K. Bongartz we get rationality of the singularities in the closures of the strata in the corresponding quiver Grassmannians.

3.1.1. Minimal Degenerations of Orbits.

Every quiver representation $X \in \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ can be written as direct sum of indecomposable nilpotent representations $U_i(\ell)$ with $\ell \leq N$. This is proven in the thesis and can also be found in [**50**, Theorem 7.6].

DEFINITION 3.6. For a nilpotent representation $U_i(\ell)$ of the equioriented cycle on n vertices the corresponding **word** $w_i(\ell)$ is defined as

$$w_i(\ell) := i \quad i+1 \quad i+2 \quad \dots \quad i+\ell-2 \quad i+\ell-1$$

where we view each number in $\mathbb{Z}_n$.

To each quiver representation X we assign a diagram

$$\vartheta_X \;=\; \begin{matrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{matrix}$$

consisting of words $w_1, \dots, w_r$ corresponding to the indecomposable summands in the decomposition of X with the same order as for the summands of X.

REMARK. By convention we always write the words in a diagram such that the last letters of the words are in one column. This helps us to count repetitions of certain letters in the words starting from their end as used in Proposition 3.10.

EXAMPLE 3.7. Let $n = 4$, $X = U_1(5) \oplus U_2(4) \oplus U_3(3) \oplus U_4(2)$. Then

$$\vartheta_X = \begin{array}{r} 12341 \\ 2341 \\ 341 \\ 41 \end{array}$$

PROPOSITION 3.8 (p. 28). Two elements of $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ are conjugate under the action of $G := \mathrm{GL}_{\mathbf{e}}$ if and only if their diagrams are the same with respect to permutations of the words.

Let X and Y be elements of $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ with diagrams ϑ_X and ϑ_Y. If $G.Y$ is included in the closure $\overline{G.X}$, we write $\vartheta_Y \leq \vartheta_X$. We call Y and X (resp. ϑ_Y and ϑ_X) **adjacent** if $G.Y \subset \overline{G.X}$ and there exists no $Z \in \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ with $G.Y \subsetneq \overline{G.Z} \subsetneq \overline{G.X}$, i.e. the orbit $G.Y$ is dense in $\overline{G.X} \setminus G.X$. For adjacent X and Y we call $G.Y \subset \overline{G.X}$ (resp. $\vartheta_Y < \vartheta_X$) a **minimal degeneration**.

The subsequent proposition characterises the orbits and orbit closures in the variety of nilpotent representations. For $i \in \mathbb{Z}_n$, $j \in \mathbb{Z}_N$ and $U \in \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ define

$$U_{\alpha_i}^{\circ j} := U_{\alpha_{i+j}} \circ U_{\alpha_{i+j-1}} \circ \cdots \circ U_{\alpha_{i+2}} \circ U_{\alpha_{i+1}} \circ U_{\alpha_i}.$$

PROPOSITION 3.9 (p. 32). The orbit closure of the nilpotent representation $X \in \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ is given as

$$\overline{G.X} = \left\{ Y \in \mathrm{R}_{\mathbf{e}}(\Delta_n) : \ \operatorname{corank} Y_{\alpha_i}^{\circ j} \ \geq \ \operatorname{corank} X_{\alpha_i}^{\circ j} \quad \text{for all } i \in \mathbb{Z}_n, j \in \mathbb{Z}_N \right\}.$$

In particular $\overline{G.X}$ contains only nilpotent representations of Δ_n.

For a representation $X \in \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ let $x_p(i)$ be the number of repetitions of the letter $i \in \mathbb{Z}_n$ in the last p columns of the diagram ϑ_X.

PROPOSITION 3.10 (p. 69). Let $X, Y \in \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$. Then

$$G.Y \subseteq \overline{G.X} \iff y_p(i) \geq x_p(i) \text{ for all } i \in \mathbb{Z}_n \text{ and all } p \in [N].$$

For two words w_1 and w_2 in the letters $\mathbb{Z}_n$, we can build the word $w_1 w_2$ if w_1 ends with i and w_2 starts with $i + 1$.

EXAMPLE 3.11. Let $n = 4$ and consider the words $w_1 = 23412$ and $w_2 = 341234$ which correspond to the representations $U_2(5)$ and $U_3(6)$. It is possible to build

$$w_1 w_2 = 23412341234$$

which corresponds to the indecomposable representation $U_2(11)$ but $w_2 w_1$ does not exist.

THEOREM 3.12 (p. 73). *For a minimal degeneration $\vartheta_Y < \vartheta_X$ it is possible to obtain the diagram of Y from the diagram of X by replacing a pair of words w, w' in ϑ_X of the form $w = w_1 w_2 w_3$, $w' = w_2$ by the pair $w_2 w_3$, $w_1 w_2$. Here the w_i satisfy one of the following conditions:*

A) $|w_3| < n$ *and* $|w_1| \geq |w_3|$,

B) $|w_1| < n$ *and* $|w_3| \geq |w_1|$,

C) $|w_1| = n$ *and* $|w_3| = r \cdot n$ *for* $r \geq 1$.

REMARK. This result can also be used to describe the degeneration starting from Y. Thus it is possible to determine the orbit structure of $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ starting from the orbit of $S_{\mathbf{e}}$ which is zero-dimensional and is included in every orbit closure of nilpotent representations.

Moreover we can describe the structure of the stratification of a quiver Grassmannian for Δ_n if we start with a representative of the stratum of smallest dimension and use that by Theorem 2.3 the representatives of closures of strata and closures of orbits coincide. This stratum can be determined using Lemma 4.19.

THEOREM 3.13 (p. 96). *Let $\vartheta_Y < \vartheta_X$ be a minimal degeneration of type A,B or C from Theorem 3.12 and let e denote the number of words in ϑ_Y which are of the form w_1w_2 or w_2w_3. Then*

$$\operatorname{codim}_{\overline{G.X}} G.Y = \begin{cases} e-1 & \text{in case } A \text{ and } B \\ 2(e-1) & \text{in case } C \end{cases}$$

REMARK. The results in Section 3.2 about the Hom-space dimensions allow us to use this theorem to compute the codimension in the closures of the strata.

3.1.2. Singularities in Minimal Degenerations and in Codimension 2. Let U and W be two varieties with $u \in U$ and $w \in W$. The singularity of U in u is called **smooth equivalent** to the singularity of W in w if there exists a variety Z, a $z \in Z$ and two morphisms

$$f : Z \to U \text{ and } g : Z \to W$$

such that $f(z) = u$, $g(z) = w$ and f and g are smooth [**1**], [**43**].

This defines an equivalence class of punctured varieties (U, u) which we denote by $\operatorname{Sing}(U, u)$. For the action of an regular algebraic group on U we get

$$\operatorname{Sing}(U, u) = \operatorname{Sing}(U, u')$$

if u and u' belong to the same orbit $\mathcal{O}$. In this case $\operatorname{Sing}(U, \mathcal{O})$ denotes the equivalence class.

Let U and W be vector spaces of dimension p respective q and define

$$\mathrm{D}_{p,q} := \big\{X \in \operatorname{Hom}(U, W) : \operatorname{rank} X = 1\big\}.$$

For the closure we obtain

$$\overline{\mathrm{D}_{p,q}} = \mathrm{D}_{p,q} \cup \{0\}$$

and 0 is an isolated rational singularity in $\overline{\mathrm{D}_{p,q}}$ [**48**, Section 4.5.], [**47**, §3]. We denote this equivalence class by $d_{p,q}$, i.e.

$$d_{p,q} := \operatorname{Sing}\big(\overline{\mathrm{D}_{p,q}}, 0\big).$$

In $\operatorname{End}(U)$ there exists a uniquely determined nilpotent conjugation class C of minimal dimension $2(p-1)$ [**48**, Section 5.9.c)].

The closure of C is $\overline{C} = C \cup \{0\}$ and 0 is an isolated singularity in $\overline{C}$ [**52**, 2.3]. We denote this equivalence class by a_{p-1}, i.e.

$$a_{p-1} := \operatorname{Sing}(\overline{C}, 0).$$

Let N be the set including all nilpotent elements of $\operatorname{End}(U)$. There is a unique nilpotent orbit of codimension 2 in N, the so called subregular orbit $\mathcal{O}_s$. By a result of E. Brieskorn we get

$$\operatorname{Sing}(\overline{N}, \mathcal{O}_s) = A_{p-1}$$

where $A_{p-1} := \operatorname{Sing}(Z, 0)$ is the singularity of the surface

$$Z := \big\{(x, y, z) \in \mathbb{C}^3 : x^p + yz = 0\big\}$$

in the origin [**12**].

We have the following relations on the types of singularities:

$$A_1 = a_1,\ d_{p,q} = d_{q,p}$$

and $\overline{\mathrm{D}_{p,q}}$ is smooth for $p = 1$ or $q = 1$.

Now we want to determine $\mathrm{Sing}(\overline{G.X}, G.Y)$ explicitly. For type A and B we define p as the number of words of the form w_1w_2 in ϑ_Y and q as the number of words of the form w_2w_3 in ϑ_Y.

For type C we distinguish the cases:
C1): $|w_1| = |w_3| = n$, i.e. $w_1w_2 = w_2w_3$. Here we set ℓ as the number of words of the form w_1w_2 in ϑ_Y.
C2): $|w_1| = n$ and $|w_3| = r \cdot n$ for $r \geq 1$. We set $\ell := r + 1$.

THEOREM 3.14 (p. 114). *Let X and Y be adjacent elements of $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$. Then*

$$\mathrm{Sing}\big(\overline{G.X}, G.Y\big) = \begin{cases} d_{p,q} & \text{in case } A \text{ and } B \\ a_{\ell-1} & \text{in case } C1 \\ A_{\ell-1} & \text{in case } C2. \end{cases}$$

THEOREM 3.15 (p. 132). *Let X and Y be adjacent elements of $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ and $\mathrm{codim}_{\overline{G.X}}\, G.Y = 2$. Then $\overline{G.X}$ is smooth in $G.Y$ or $\mathrm{Sing}(\overline{G.X}, G.Y) = A_\ell$ for some $\ell \in \mathbb{N}$.*

This ℓ is determined uniquely and the closures of the orbits are smooth in codim ≤ 2 if $e_i = 0$ for some $i \in \mathbb{Z}_n$. If $\mathrm{codim}_{\overline{G.X}}\, G.Y = 1$, the closure $\overline{G.X}$ is smooth in the orbit $G.Y$ because $\overline{G.X}$ is normal [**69**, II, §5, Theorem 5].

3.2. Morphisms of Quiver Representations and Words

In this subsection we develop the main tools required to prove the parametrisation of the irreducible components for the class of quiver Grassmannians for the equioriented cycle as introduced in the beginning of this chapter.

Recall that to each indecomposable nilpotent representation of the equioriented cycle there is assigned a word with letters in $\mathbb{Z}_n$. Define $r_j(w)$ as the number of repetitions of the letter j in a word w. These numbers can be used to compute the dimension of the space of morphisms between two indecomposable nilpotent representations of the cycle. The linearity of the morphisms allows us to generalise this formula to compute the dimension of the morphism space for all nilpotent representations of the cycle.

PROPOSITION 3.16. For two indecomposable nilpotent representations $U_i(\ell)$ and $U_j(k)$ of Δ_n let $w_i(\ell)$ and $w_j(k)$ be the corresponding words. Then the dimension of the space of morphisms from $U_i(\ell)$ to $U_j(k)$ is computed as

$$\begin{aligned} \dim \mathrm{Hom}_{\Delta_n}\big(U_i(\ell), U_j(k)\big) &= \min\big\{r_i\big(w_j(k)\big), r_{j+k-1}\big(w_i(\ell)\big)\big\} \\ &= r_{j+k-1}\big(w_i(m)\big). \end{aligned}$$

where $m := \min\{\ell, k\}$.

This proposition is just a different formulation of the subsequent theorem by A. Hubery [**44**, Theorem 16 (1)].

Theorem 3.17. *Let $M_{(i;\ell)}$ and $M_{(j;m)}$ be two indecomposable nilpotent representations of the equioriented cycle. The dimension of the space of morphisms from $M_{(i;\ell)}$ to $M_{(j;m)}$ is computed as*

$$\dim \operatorname{Hom}_{\Delta_n}\big(M_{(i;\ell)}, M_{(j;m)}\big) = \big|\big\{ \max\{0, \ell - m\} \leq r \leq \ell - 1 : r \equiv j - i \mod n\big\}\big|.$$

In this statement the orientation of the quiver is the other way around, i.e. the index of the vertices decreases by one along an arrow whereas in this work it increases by one.

Moreover the indecomposable nilpotent representations have an injective labelling, i.e. $M_{(j;N)} = I_j$ in $\operatorname{rep}_{\Bbbk}(\Delta_n, \mathrm{I}_N)$ where in this work we label the indecomposable nilpotent representations $U_i(\ell)$ projective such that $U_i(N) = P_i$ in $\operatorname{rep}_{\Bbbk}(\Delta_n, \mathrm{I}_N)$.

For some applications it is useful to work with the injective labelling of representations of the equioriented cycle. This is the case if we look at embeddings of indecomposable representations or successor closed subquivers of the coefficient quiver of a quiver representations as done in Chapter 4. Hence we define

$$U(j;N) := U_{j-N+1}(N).$$

Proof of Proposition 3.16. Translating the theorem of Andrew Hubery into our notation we obtain

$$\dim \operatorname{Hom}_{\Delta_n}\big(U_i(\ell), U_j(m)\big) = \big|\big\{0 \leq r \leq \min\{\ell, m\} - 1 : r \equiv j - i + m - 1 \mod n\big\}\big|.$$

This holds since both sets count the possibilities to write the words corresponding to the source and target representation of the morphisms parallel such that in the overlap both words have the same letters, the word of the source representation does not start before the word of the target representation and the word of the target representation does not end after the word of the source representation.

In our notation this equals the number of repetitions of the letter $j + m - 1$ in the word corresponding to $U_i(k)$ with $k = \min\{\ell, m\}$ where we have to use the minimum to ensure that the word of the source representation $U_i(\ell)$ does not start before the word of the target representation $U_j(m)$.

This can also be done by taking the minimum of the repetitions of the letter $j + m - 1$ in the word corresponding to $U_i(\ell)$ and the repetitions of the letter i in the word corresponding to $U_j(m)$ such that both, the ending and the starting condition are handled in the same way. □

It is also possible to compute the dimension of the space of morphisms by counting certain repetitions of the letter i in the word corresponding to $U_j(k)$. Here we have to exclude the repetitions coming before $\max\{0, m - \ell\}$ such that the parametrisation of the word wherein we have to count the repetitions becomes more complicated and we exclude this case from the proposition.

Definition 3.18. An embedding of quiver representations $\varphi : U \hookrightarrow V$ is called **decomposable** if there are embeddings

$$\psi_1 : U_1 \hookrightarrow V_1 \quad \text{and} \quad \psi_2 : U_2 \hookrightarrow V_2$$

such that $U = U_1 \oplus U_2$ holds and for the decomposition $V = V_1 \oplus V_2$ both summands are non-zero. An embedding which is not decomposable is called **indecomposable**.

This does not imply that the original embedding φ equals the embedding of the decompositions

$$\psi = (\psi_1, \psi_2) : U_1 \oplus U_2 \hookrightarrow V_1 \oplus V_2.$$

PROPOSITION 3.19. In the category $\mathrm{rep}_{\Bbbk}(\Delta_n, \mathrm{I}_N)$ all indecomposable embeddings of representations are of the form

$$\varphi : U_i(\ell) \hookrightarrow U_{i-k}(\ell + k)$$

for $k \in \mathbb{Z}_{\geq 0}$.

REMARK. With the notation using an injective labelling of indecomposable representations, the indecomposable embeddings are of the form

$$\varphi : U(j; \ell) \hookrightarrow U(j; \ell + k)$$

for $k \in \mathbb{Z}_{\geq 0}$.

This result about the structure of the embeddings of quiver representations for Δ_n helps us to examine the structure of the corresponding quiver Grassmannians. For the proof we require some statements about the structure of coefficient quivers which will be introduced in Chapter 4. This section is independent of the current section and therein we give a proof of this proposition using Corollary 4.12.

It is also possible to prove this statement independent of the results about torus fixed points. For that independent proof we require information about the structure of Auslander Reiten quivers for Δ_n. But since we do not need it elsewhere, we decide to omit their construction and the results about their shape from this thesis. The structure of Auslander Reiten quivers for Δ_n is very similar to the structure of Auslander Reiten quivers for equioriented quivers of type A.

We can use the word combinatorics to compute the dimension of the space of morphisms from an arbitrary representation in $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ to an indecomposable representation of maximal length in this variety of quiver representations.

PROPOSITION 3.20. Let $M \in \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ then

$$\dim \mathrm{Hom}_{\Delta_n}\big(M, U_i(N)\big) = e_{i+N-1} \text{ for all } i \in \mathbb{Z}_n.$$

REMARK. Here it is important that the indecomposable representation $U_i(N)$ is of maximal length or at least longer than every summand of M. Otherwise M could contain $U_{i-1}(N+1)$ which contradicts the statement of the proposition because

$$\dim \mathrm{Hom}_{\Delta_n}\big(U_{i-1}(N+1), U_i(N)\big) = 0$$

by Proposition 3.16 since $r_{i+N}\big(w_i(N)\big) = 0$. For the injective labelling of the indecomposable representations the statement of the proposition reads as

$$\dim \mathrm{Hom}_{\Delta_n}\big(M, U(j; N)\big) = e_j \text{ for all } j \in \mathbb{Z}_n.$$

PROOF OF PROPOSITION 3.20. Every representation $M \in \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ is nilpotent and can be written as direct sum of indecomposable nilpotent representations, namely

$$M \cong \bigoplus_{j \in \mathbb{Z}_n} \bigoplus_{\ell \in [N]} U_j(\ell) \otimes \Bbbk^{m_{j,\ell}}.$$

Thus

$$\dim \mathrm{Hom}_{\Delta_n}\big(M, U_i(N)\big) = \sum_{j \in \mathbb{Z}_n} \sum_{\ell \in [N]} m_{j,\ell} \cdot \dim \mathrm{Hom}_{\Delta_n}\big(U_j(\ell), U_i(N)\big)$$

because the morphisms of quiver representations are linear.

Now it suffices to show

$$\dim \operatorname{Hom}_{\Delta_n}\big(U_j(\ell), U_i(N)\big) = \big(\mathbf{dim}\, U_j(\ell)\big)_{i+N-1}.$$

We want to apply Proposition 3.16. By assumption we know that $\ell \leq N$ and hence we have to count the repetitions of the vertex $i+N-1$ (which is the end point of $U_i(N)$) in the word $w_j(\ell)$ corresponding to $U_j(\ell)$. This value is given by the $(i+N-1)$-th entry of the dimension vector of $U_j(\ell)$. □

Proposition 3.20 implies that the codimension of the orbits in $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ equals the codimension of the strata in the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$. Accordingly we can compute the orbit dimensions in $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ for the elements of the Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ in order to find the strata of highest dimension in the quiver Grassmannian. By Lemma 1.3 the dimension of the stratum of $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ is given as

$$\dim \operatorname{Hom}_{\Delta_n}\big(U, X \oplus Y\big) - \dim \operatorname{End}_{\Delta_n}\big(U\big).$$

For the first part we can apply Proposition 3.20 and obtain

$$\dim \operatorname{Hom}_{\Delta_n}\big(U, X \oplus Y\big) = \dim \operatorname{Hom}_{\Delta_n}\big(X, X \oplus Y\big)$$

for all $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ because X and Y consist of summands of the form $U_i(N)$ and the morphisms are linear. Hence we are interested in the value of

$$\dim \operatorname{End}_{\Delta_n}\big(U\big) = \dim \operatorname{Hom}_{\Delta_n}\big(U, U\big)$$

and want to find the elements of the quiver Grassmannian minimising this value. This is equal to the codimension of an orbit in the variety of quiver representations because for $U \in \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ we compute its orbit dimension as

$$\dim \mathrm{GL}_{\mathbf{e}}.U = \dim \mathrm{GL}_{\mathbf{e}} - \dim \operatorname{Hom}_{\Delta_n}\big(U, U\big).$$

For the rest of this section, we turn our attention to the dimensions of the orbits in $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ for elements of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$.

Remark. Let Q be a Dynkin quiver and let X, Y be exceptional representations of Q such that $\mathrm{Ext}_Q^1(X, Y) = 0$. Then

$$\dim \operatorname{Hom}_Q\big(U, U\big) \geq \dim \operatorname{Hom}_Q\big(X, X\big)$$

holds for all $U \in \mathrm{Gr}_{\mathbf{e}}^{Q}(X \oplus Y)$, i.e. the dimension of the orbit of X in $\mathrm{R}_{\mathbf{e}}(Q)$ is the highest among all elements of the quiver Grassmannian. Thus we obtain

$$\dim \mathrm{Gr}_{\mathbf{e}}^{Q}(X \oplus Y) = \dim \operatorname{Hom}_Q\big(X, Y\big)$$

and the quiver Grassmannian is the closure of the stratum of X. For more details on this see Section 3.1 in [**20**].

Unfortunately this does not hold in full generality for the equioriented cycle.

Proposition 3.21. Let $N = \omega \cdot n$ for $\omega \in \mathbb{N}$. Then

$$\dim \operatorname{Hom}_{\Delta_n}\big(U, U\big) \geq \dim \operatorname{Hom}_{\Delta_n}\big(X, X\big)$$

holds for all $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$.

There are counter examples for this result if $N \neq \omega \cdot n$. To carry out the computations in the examples is lengthy such that we put them in the appendix to this thesis (see Example A.2 and Example A.3).

The indecomposable representation $U_i(\omega n)$ has length ωn which means that in the picture of Proposition 3.2 it is winding around the cycle on n points exactly ω times. For this reason we refer to ω as winding number.

PROOF. The idea of the proof is to show that for every element $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ its orbit in the variety of quiver representations degenerates to an orbit with the same codimension as the orbit of X or already has the same codimension. In the first case it follows from the definition of the degeneration of orbits that the codimension of its orbit is strictly bigger. The subsequent property of a subrepresentation helps us to decide how far the representation is degenerate from X.

For $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ define the set

$$S(U) := \big\{ U_i(\ell) \subseteq U \text{ summand } : \ell < N \big\}$$

including all direct summands of U which are not of maximal length. If the set $S(U)$ is empty, we directly obtain

$$\dim \mathrm{Hom}_{\Delta_n}\big(U, U\big) = \dim \mathrm{Hom}_{\Delta_n}\big(X, X\big)$$

since

$$\dim \mathrm{Hom}_{\Delta_n}\big(U_i(N), U_j(N)\big) = \omega$$

for all $i, j \in \mathbb{Z}_n$ and $N = \omega \cdot n$. This follows from Proposition 3.20 if we set $M = U_i(N)$.

Now let $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ be given such that $S(U) \neq \emptyset$. Since all entries of the dimension vector $\mathbf{dim}\, U$ are equal and it is divided into segments of length at most N, $S(U)$ has to contain at least two elements.

We can find $U_i(\ell), U_j(k) \in S(U)$ and can assume without loss of generality that j is contained in the word $w_i(\ell+1)$. This pair has to exist because otherwise the dimension vector of U could not be homogeneous, i.e. all entries are equal. By changing the labelling of the two representations we can ensure that they satisfy the relation we want.

Let w_2 be the overlap of the words $w_i(\ell)$ and $w_j(k)$. We can write them as $w_i(\ell) = w_1 w_2$ and $w_j(k) = w_2 w_3$ where it is possible that w_2 is the empty word.

We define

$$\hat{U} := U \oplus U_i(i - j + \ell + k) \oplus U_j(j - i) \setminus U_i(\ell) \setminus U_j(k).$$

This representation $\hat{U}$ embeds into the same summands of $X \oplus Y$ as U did. By Proposition 3.19 we know that $U_i(i - j + \ell + k)$ embeds into the same $U_p(N)$ as $U_j(k)$ and $U_j(j-i)$ embeds into the same $U_q(N)$ as $U_i(\ell)$. Moreover U and $\hat{U}$ have the same dimension vector. It remains to show that $\dim \mathcal{O}_U < \dim \mathcal{O}_{\hat{U}}$.

In the terminology of words the representation $U_i(i-j+\ell+k)$ corresponds to $w_1w_2w_3$ and $U_j(j-i)$ corresponds to w_2. We can assume that the word $w_1w_2w_3$ has not more than N letters because there have to exist $w_i(\ell) = w_1w_2$ and $w_j(k) = w_2w_3$ such that this is satisfied. Without such words it would not be possible that the dimension vector of U is homogeneous and that all words corresponding to it have at most N letters.

Following Theorem 3.12 the orbit of U degenerates to the orbit of $\hat{U}$. Hence we get $\mathcal{O}_U \subset \overline{\mathcal{O}_{\hat{U}}}$ and $\dim \mathcal{O}_U < \dim \mathcal{O}_{\hat{U}}$. This degeneration might not be minimal but here it is not of interest to find minimal degenerations. Thus we do not have to satisfy the restrictions on the words in the theorem.

Since all the vector spaces U_i for $i \in \mathbb{Z}_n$ corresponding to the subrepresentation U are equidimensional we can apply this procedure starting from any $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ until we arrive at an $\hat{U} \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ with $S(\hat{U}) = \emptyset$. Thus we obtain

$$\dim \mathrm{Hom}_{\Delta_n}\big(U, U\big) > \dim \mathrm{Hom}_{\Delta_n}\big(X, X\big)$$

for every $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ with $S(U) \neq \emptyset$. □

For the proof it is crucial that the dimension vector of the subrepresentations is homogeneous and that the length of the cycle divides the length of the indecomposable projective and injective representations. This is guaranteed by the condition $N = \omega n$. Otherwise we can not assure that the gluing procedure of the words ends in a representation with $S(U) = \emptyset$. In this setting it is not possible to control the minimal codimension of the orbits which could be obtained as explained in Example A.2 and Example A.3.

3.3. Irreducible Components of the Quiver Grassmannian for $N = \omega \cdot n$

For the remainder of this chapter we restrict us to the case $N = \omega n$. The bounded projective and injective representations in $\mathrm{rep}_{\Bbbk}(\Delta_n, \mathrm{I}_{\omega n})$ will be denoted by P_i^{ω} and I_j^{ω}. From the proof of Proposition 3.21 it follows not only that all subrepresentations in $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ have bigger or equal codimension than X. We even get a characterisation of the subrepresentations with the same codimension as X. Namely they are parametrised by the condition $S(U) = \emptyset$. Based on this observation we can determine the dimension and the irreducible components of the quiver Grassmannian which we already know to be equidimensional. The dimension of the quiver Grassmannian is given by the dimension of the stratum of X_ω and is computed below.

Lemma 3.22. *Let*

$$X_\omega := \bigoplus_{i \in \mathbb{Z}_n} P_i^{\omega} \otimes \Bbbk^{x_i} \text{ and } Y_\omega := \bigoplus_{j \in \mathbb{Z}_n} I_j^{\omega} \otimes \Bbbk^{y_j}$$

and set $\mathbf{e}_\omega := \mathbf{dim}\, X_\omega$, *where* $x_i, y_j \in \mathbb{N}$ *for all* $i, j \in \mathbb{Z}_n$. *The dimension of the quiver Grassmannian is computed as*

$$\dim \mathrm{Gr}_{\mathbf{e}_\omega}^{\Delta_n}(X_\omega \oplus Y_\omega) = \omega k(m - k),$$

where $k := \sum_{i \in \mathbb{Z}_n} x_i$ *and* $m := \sum_{i \in Z_n} x_i + y_i$.

Proof. From Proposition 3.21 and Proposition 3.20 and Lemma 1.3 it follows that

$$\dim \mathcal{S}_U \leq \dim \mathcal{S}_{X_\omega}$$

holds for every $U \in \mathrm{Gr}_{\mathbf{e}_\omega}^{\Delta_n}(X_\omega \oplus Y_\omega)$. This inequality does not hold for arbitrary N as Example A.2 and Example A.3 show.

It remains to compute the dimension of the stratum of X_ω which is given by

$$\begin{aligned}\dim \mathcal{S}_{X_\omega} &= \dim\ \mathrm{Hom}_{\Delta_n}\big(X_\omega, X_\omega \oplus Y_\omega\big) - \dim\ \mathrm{Hom}_{\Delta_n}\big(X_\omega, X_\omega\big)\\ &= \dim\ \mathrm{Hom}_{\Delta_n}\big(X_\omega, Y_\omega\big).\end{aligned}$$

Applying Proposition 3.20 we obtain

$$\dim \mathrm{Hom}_{\Delta_n}\big(U_i(\omega n), U_j(\omega n)\big) = \big(\mathbf{dim}\, U_i(\omega n)\big)_{j+\omega n-1} = \omega$$

for all $i, j \in \mathbb{Z}_n$ and compute the dimension of the stratum explicitly as

$$\begin{aligned}\dim \mathrm{Hom}_{\Delta_n}\big(X_\omega, Y_\omega\big) &= \sum_{i\in\mathbb{Z}_n}\sum_{j\in\mathbb{Z}_n} \dim \mathrm{Hom}_{\Delta_n}\big(P_i^\omega \otimes \Bbbk^{x_i}, I_j^\omega \otimes \Bbbk^{y_j}\big)\\ &= \sum_{i\in\mathbb{Z}_n} x_i \sum_{j\in\mathbb{Z}_n} y_j \dim \mathrm{Hom}_{\Delta_n}\big(P_i^\omega, I_j^\omega\big) = \sum_{i\in\mathbb{Z}_n} x_i \sum_{j\in\mathbb{Z}_n} y_j \cdot \omega\\ &= \sum_{i\in\mathbb{Z}_n} x_i \cdot \omega \cdot (m-k) = \omega \cdot k(m-k)\end{aligned}$$

where $P_i^\omega \cong U_i(\omega n)$ and $I_j^\omega \cong U_{j-\omega n+1}(\omega n)$. □

Based on the characterisation of the strata with the same codimension as the stratum of X_ω from the previous section we obtain the subsequent parametrisation of the irreducible components of the quiver Grassmannians.

Lemma 3.23. *The irreducible components of* $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega)$ *are in bijection with the set*

$$C_k(\mathbf{d}) := \Big\{\mathbf{p} \in \mathbb{Z}^n_{\geq 0} : p_i \leq d_i \text{ for all } i \in \mathbb{Z}_n, \sum_{i\in\mathbb{Z}_n} p_i = k\Big\},$$

where $d_i := y_i + x_{i+1}$ *and they all have dimension* $\omega k(m-k)$.

Remark. In particular, the number of irreducible components is independent of the winding number ω.

Proof. We use the interpretation of the Grassmannian as framed moduli space

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega) \cong \mathrm{R}^s_{\mathbf{e}_\omega,\mathbf{d}}(\Delta_n, \mathrm{I}_{\omega n})/\mathrm{GL}_{\mathbf{e}_\omega}.$$

Here the entiries of $\mathbf{d}$ are given by the multiplicities of the injective representations I_j^ω as summand of $X_\omega \oplus Y_\omega$ and these numbers are independent of the winding number ω. We can apply Theorem 2.3 to this setting such that the irreducible components of $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega)$ which are given by the closures of some strata are in bijection with the maximal elements of $\mathrm{R}^{(\mathbf{d})}_{\mathbf{e}_\omega}(\Delta_n, \mathrm{I}_{\omega n})/\mathrm{GL}_{\mathbf{e}_\omega}$ with respect to the partial order induced by the inclusion of orbit closures as introduced in Section 3.1.1.

In the proof of Proposition 3.21 we have seen that the maximal elements for this order are given by the $U \in \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega)$ such that $S(U) = \emptyset$, i.e. all summands of U have the same dimension vector and are of the form $U_i(\omega n)$ for some $i \in \mathbb{Z}_n$. The set of all such subrepresentations of $X_\omega \oplus Y_\omega$ with dimension vector $\mathbf{e}_\omega$ is parametrised by the set $C_k(\mathbf{d})$. For every tuple $\mathbf{p} \in C_k(\mathbf{d})$ define the representation

$$U(\mathbf{p}) := \bigoplus_{i\in\mathbb{Z}_n} U_{i+1}(N) \otimes \Bbbk^{p_i}.$$

We get $i+1$ as index since d_i is the multiplicity of I_i which is isomorphic to $U_{i-N+1}(N) = U_{i+1}(N)$ for $N = \omega \cdot n$. The assumption on the summation

$$\sum_{i\in\mathbb{Z}_n} p_i = k$$

ensures that the dimension vector of $U(\mathbf{p})$ is $\mathbf{e}$. The restriction

$$p_i \in \{0, 1, \ldots, d_i\} \text{ for all } i \in \mathbb{Z}_n$$

guarantees that the representation $U(\mathbf{p})$ corresponding to the tuple $\mathbf{p}$ embeds into $X_\omega \oplus Y_\omega$. The dimension of the irreducible components is computed in the same way as done for the stratum $\mathcal{S}_{X_\omega}$ in Lemma 3.22. □

For arbitrary N we would have $d_i := y_i + x_{i-N+1}$ but for $N = \omega \cdot n$ the shift by N does not change the index because it is considered as a number in $\mathbb{Z}_n$. The number of irreducible components is bounded by $\binom{n+k-1}{k}$ since

$$C_k(\mathbf{d}) \subseteq C_k := \Big\{\mathbf{p} \in \mathbb{Z}^n_{\geq 0} : p_i \leq k \text{ for all } i \in \mathbb{Z}_n, \sum_{i \in Z_n} p_i = k\Big\}.$$

The set C_k contains all partitions of the number k into at most n parts. This is equivalent to choosing $n-1$ points out of $n+k-1$ points to be the separators between the n parts of a partition of the remaining k points. The number of these choices is given by the binomial coefficient

$$|C_k| = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}.$$

For the case $N = n$, this parametrisation of the irreducible components together with a precise count is proven in the thesis of N. Haupt [**41**, Proposition 3.6.16]. Because we have discovered above that the number of irreducible components is independent of the winding number ω we can use his formula for the case $\omega = 1$ to compute the number of irreducible components of the quiver Grassmannian

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}\big(X_\omega \oplus Y_\omega\big)$$

for arbitrary winding numbers ω.

3.4. Geometric Properties of the Quiver Grassmannian

Back in the setting where the indecomposable summands of X and Y have arbitrary but all the same length N we do not have a parametrisation of the irreducible components but nevertheless we get the subsequent properties.

LEMMA 3.24. *The irreducible components of* $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(X \oplus Y)$ *are normal, Cohen-Macaulay and have rational singularities.*

PROOF. In her thesis G. Kempken shows that the orbit closures inside $\mathrm{R}_{\mathbf{e}}(\Delta_n)$ are normal, Cohen-Macaulay and have rational singularities (compare Theorem 3.5 combined with Theorem 3.4). Her result holds for arbitrary representations of Δ_n. We can apply it to orbit closures of nilpotent representations in $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ because by Proposition 3.10 there are no non-nilpotent representations inside these orbit closures. Combining this with Theorem 2.3 by K. Bongartz we get that the closures of the strata in the quiver Grassmannian have rational singularities which again combined with Theorem 3.4 yields that they are normal and Cohen-Macaulay. Applying it to the highest dimensional strata we obtain the desired result. □

Moreover G. Kempken gives a description of the types of singularities which can occur and we recall it in Section 3.1.2. She also describes the structure of the orbit closures and the codimension of the minimal degenerations of orbits as summarised in Section 3.1.1.

3.5. Image in the Variety of Quiver Representations

For any set of relations I_N as defined in the beginning of Chapter 3, we have seen that every projective bounded representation of Δ_n is isomorphic to an injective bounded representation. Hence the class of quiver Grassmannians we introduced in Chapter 3 satisfies the assumptions of Theorem 2.3 such that the image of the projection

$$pr : \mathrm{R}^s_{\mathbf{e},\mathbf{d}}(\Delta_n, \mathrm{I}_N) \to \mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$$

which is denoted by $\mathrm{R}^{(\mathbf{d})}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ carries all information about the structure of the stratification of the quiver Grassmannian

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}}\big(J\big)$$

with

$$J := \bigoplus_{i \in \mathbb{Z}_n} U(i;N) \otimes \mathbb{C}^{d_i}.$$

In this section we describe two different possibilities to parametrise this image explicitly. One arises from the special shape of the bounding relations and the criteria for non-emptiness of the framed moduli space. The other is based on the $\mathbb{C}^*$-action on the quiver representation and the resulting characterisation of subrepresentations as successor closed subquivers.

3.5.1. Parametrisation by Relations and Morphisms. By Corollary 2.26 we know that there exists an embedding of a quiver representation U into the representation J if and only if

$$\dim \mathrm{Hom}_{\Delta_n}\big(S_i, U\big) \leq d_i \text{ for all } i \in \mathbb{Z}_n.$$

For the description of the image the subsequent formulation is more suitable.

Proposition 3.25. Let U be a nilpotent representation of the equioriented cycle Δ_n. Then

$$\dim \mathrm{Hom}_{\Delta_n}\big(S_i, U\big) \leq d_i \text{ for all } i \in \mathbb{Z}_n$$

if and only if

$$\dim \ker U_{\alpha_i} \leq d_i \text{ for all } i \in \mathbb{Z}_n.$$

Proof. Every nilpotent representation of Δ_n is isomorphic to a direct sum of indecomposable nilpotent representations $U(j;\ell)$ of the cycle. The dimension of the space of homomorphisms from a simple representation $S_i \in \mathrm{rep}(\Delta_n)$ to an indecomposable nilpotent representation $U(j;\ell)$ is given as

$$\mathrm{Hom}_{\Delta_n}\big(S_i, U(j;\ell)\big) = \delta_{i,j}.$$

Hence

$$\dim \mathrm{Hom}_{\Delta_n}\big(S_i, U\big) \leq d_i \text{ for all } i \in \mathbb{Z}_n$$

is satisfied if and only if U contains at most d_i indecomposable summands $U(i;\ell)$ ending at the vertex i. This is independent of the lengths ℓ of the summands. The map along the arrow $\alpha_i : i \to i+1$ corresponding to the representation $U(i;\ell)$ has a one-dimensional kernel and the kernel is zero for the all maps along the other maps of the quiver. Hence the dimension of the kernel of the map U_{α_i} matches the number of summands of U ending at the vertex i. □

For a linear map, the dimension of the kernel equals the co-rank of the matrix U_{α_i}. Thus we arrive at the equivalent condition

$$\operatorname{corank} U_{\alpha_i} \leq d_i \text{ for all } i \in \mathbb{Z}_n.$$

A representation of Δ_n satisfies all relations in I_N if and only if it satisfies the generating relations, i.e.

$$U_{\alpha_i}^{\circ N} := U_{\alpha_{i+N}} \circ U_{\alpha_{i+N-1}} \circ \cdots \circ U_{\alpha_{i+2}} \circ U_{\alpha_{i+1}} \circ U_{\alpha_i} \equiv 0 \quad \text{for all } i \in \mathbb{Z}_n.$$

Combining both we obtain the subsequent parametrisation of the image.

Proposition 3.26. For the quiver Grassmannian

$$\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}\big(J\big)$$

the image in the variety of quiver representations is parametrised as

$$\mathrm{R}_{\mathbf{e}}^{(\mathbf{d})}(\Delta_n, \mathrm{I}_N) = \Big\{ U \in \mathrm{R}_{\mathbf{e}}(\Delta_n) : \ U_{\alpha_i}^{\circ N} \equiv 0 \text{ and } \operatorname{corank} U_{\alpha_i} \leq d_i \text{ for all } i \in \mathbb{Z}_n \Big\}.$$

The orbit structure of this variety can be studied using the methods from the thesis by G. Kempken as sumariesed in Section 3.1.

3.5.2. Parametrisation by Indecomposable Representations. Every quiver representation $U \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$ is conjugated to a direct sum of indecomposable nilpotent representations, i.e.

$$U \cong \bigoplus_{i \in \mathbb{Z}_n} \bigoplus_{\ell_i = 1}^{N} U(i; \ell_i) \otimes \mathbb{C}^{d_{i,\ell_i}}$$

where $d_{i,\ell_i} \in \mathbb{Z}_{\geq 0}$ for all $i \in \mathbb{Z}_n$ and $\ell_i \in [N]$. For the space of homomorphisms we obtain

$$\dim \mathrm{Hom}_{\Delta_n}\big(S_i, U\big) = \sum_{j \in \mathbb{Z}_n} \sum_{\ell_j = 1}^{N} d_{j,\ell_j} \dim \mathrm{Hom}_{\Delta_n}\big(S_i, U(j;\ell_j)\big) = \sum_{\ell_i = 1}^{N} d_{i,\ell_i}.$$

Hence for every $U \in \mathrm{R}_{\mathbf{e}}^{(\mathbf{d})}(\Delta_n, \mathrm{I}_N)$ we can rewrite the direct sum of indecomposable representations as

$$U \cong \bigoplus_{i \in \mathbb{Z}_n} \bigoplus_{k=1}^{d_i} U(i; \ell_{i,k}) =: U(\mathbf{l})$$

where $\ell_{i,k} \in \{0, 1, 2, \ldots, N\} =: [N]_0$ for all $i \in \mathbb{Z}_n$ and all $k \in [d_i]$. Here the restriction $k \in [d_i]$ is obtained from Corollary 2.26 with the same arguments as in Section 3.5.1. For $\ell = 0$ the representation $U(i;0)$ is the zero representation which is independent of the index i. We arrive at the subsequent description.

Proposition 3.27. The $G_{\mathbf{e}}$-orbits in the variety of quiver representations $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ which correspond to strata of the quiver Grassmannian

$$\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}\big(J\big)$$

are parametrised by the set

$$\mathcal{S}_{\mathbf{e}}^{(\mathbf{d})}\big(\Delta_n, \mathrm{I}_N\big) := \Big\{ \mathbf{l} := (\ell_{i,k}) \in \bigoplus_{i \in \mathbb{Z}_n} \bigoplus_{k=1}^{d_i} [N]_0 : \ \mathbf{dim}\, U(\mathbf{l}) = \mathbf{e} \text{ and } \ell_{i,k} \geq \ell_{i,k+1} \Big\}.$$

This set is finite and can be used for the computational study of the structure of the stratification for quiver Grassmannians of the equioriented cycle. Using the result about the dimension of the space of homomorphisms from Proposition 3.16 we can define functions to compute the dimensions of orbits or strata and determine the irreducible components of the quiver Grassmannians computationally.

CHAPTER 4

Torus Action on the Quiver Grassmannian

In this chapter we introduce an action of the torus $\mathbb{C}^*$ on the quiver Grassmannians for the equioriented cycle. This allows us to compute cellular decopositions, Euler-Poincaré characteristics and Poincaré polynomials of these quiver Grassmannians. But first we want to recall the general definitions and methods we are using.

4.1. Property (S) and (C)

DEFINITION 4.1. A finite partition $(X_i)_{i\in[m]}$ of a complex algebraic variety X is said to be an α**-partition** if

$$\bigcup_{i\in[k]} X_i \text{ is closed in } X \text{ for all } k \in [m].$$

DEFINITION 4.2. A **cellular decomposition** or **affine paving** of X is an α-partition into parts X_i which are isomorphic to affine spaces.

We say X has **property (C)** if X admits a cellular decomposition. For a complex variety X, the Borel-Moore homology with integer coefficients of X equipped with the analytic topology is denoted by

$$H_i(X) := H_i^{\mathrm{BM}}\big(X(\mathbb{C}); \mathbb{Z}\big).$$

Equivalent definitions of the Borel-Moore homology can be found in the book by N. Chris and V. Ginzburg [**22**, Chapter 2.6]. If X is an algebraic variety, the group generated by k-dimensional irreducible subvarieties modulo rational equivalence is denoted by $A_k(X)$. There exists a canonical homomorphism

$$\varphi_i : A_i(X) \to H_{2i}(X)$$

which is called **cycle map** by W. Fulton [**34**].

DEFINITION 4.3. The variety X has **property (S)** if

(1) numerical and rational equivalence on X coincide,
(2) $H_i(X) = 0$ for i odd and
(3) the cycle map $\varphi_i : A_i(X) \to H_{2i}(X)$ is an isomorphism for all i.

The concept of property (S) was introduced by C. De Concini, G. Lusztig and C. Procesi in [**25**] in order to replace the notion of cellular decompositions. Indeed we have the following implication. The converse is not true in general.

LEMMA 4.4. *X has property (S) if X admits a cellular decomposition.*

PROOF. C. De Concini, G. Lusztig and C. Procesi showed that X has property (S) if it admits an α-partition into pieces having property (S) [**25**, Lemma 1.8]. Moreover they proved that for a vector bundle $E \to X$, E has property (S) if X has property (S) [**25**, Lemma 1.9]. Since X admits a cellular decomposition, we have an α-partition of X into affine spaces $(X_i)_{i\in[m]}$. Hence we can view X as vector bundle over

$$Y := \{p_i : i \in [m]\}$$

where p_i is some point in X_i. Points have property (S) and Y admits an α-partition into the sets $Y_i := \{p_i\}$. By [**25**, Lemma 1.8] Y has property (S) and it follows by [**25**, Lemma 1.9] that X as vector bundle over Y has property (S). □

In general it is a open question which class of quiver Grassmannians admit cellular decompositions. It was conjectured that quiver Grassmannians for rigid representations of acyclic quivers have property (C). This question is still open but it is shown by G. Cerulli Irelli, F. Esposito, H. Franzen and M. Reineke in [**18**] that quiver Grassmannians for Dynkin quivers have property (C) and if M is a rigid representation of an arbitrary quiver the corresponding quiver Grassmannians have property (S). In Section 4.4 we show that certain quiver Grassmannians for the equioriented cycle have property (C) and hence also property (S). This will allow us to use the combinatorics of the coefficient quiver in order to study properties of the corresponding quiver Grassmannians. In Chapter 5 and Chapter 6 this is applied to examine approximations of the affine Grassmannian and the affine flag variety.

4.2. About $\mathbb{C}^*$-Actions

Let X be a complex projective variety with an algebraic $\mathbb{C}^*$-action

$$\mathbb{C}^* \times X \to X$$
$$(z, x) \mapsto z.x.$$

Let S be the set of fixed points (stable points) of the $\mathbb{C}^*$-action on X, i.e.

$$S := \big\{x \in X : z.x = x \text{ for all } z \in \mathbb{C}^*\big\}.$$

For $p \in S$ the **attracting set** X_p is defined as

$$X_p := \big\{x \in X : \lim_{z\to 0} z.x = p\big\}.$$

If this action has finitely many fixed points, the variety X admits a decomposition into their attracting sets. For smooth varieties this is stated by N. Chris and V. Ginzburg in [**22**, Theorem 2.4.3]. The formulation we gibe below is based on the article by R. Gonzales [**38**, Theorem 4.3]. The original version of this theorem is stated in the article by A. Bialynicki-Birula [**6**].

Theorem 4.5. *Let X be a normal projective variety with a $\mathbb{C}^*$-action and a finite number of fixed points. The attracting sets of the fixed points form a disjunct decomposition*

$$X = \bigcup_{p\in S} X_p$$

and this decomposition is an α-partition.

Sometimes decompositions of this from are refered to as Bialynicki-Birula-decompositions or BB-decompositions because they first appeared in the article by A. Bialynicki-Birula [**5**].

Corollary 4.6. If the attracting sets X_p are affine varieties, a $\mathbb{C}^*$-action on X with finitely many fixed points implies that X admits a cellular decomposition.

For any topological space X the i-th Betti number b_i is defined as the rank of the i-th singular homology group with integer coefficients $H_i(X, \mathbb{Z})$, if this group is finitely generated [**70**, p.176]. The Euler-Poincaré characteristic of X is defined as the alternating sum

$$\chi(X) := \sum_i (-1)^i b_i,$$

if there are only finitly many non-zero Betti numbers. If the topological space X admits a finite cellular decomposition the Euler-Poincaré characteristic of X equals the alternating sum

$$\chi(X) = \sum_k (-1)^k c_k,$$

where c_k is the number of k-dimensional cells over the ground field $\mathbb{R}$ [**40**, Chapter 2.2.2]. Let The Poincaré series of X is the generating function of the Betti numbers of X. If X has a cellular decomposition it equals the sum

$$p_X(t) = \sum_{k=0}^{\infty} c_k t^k$$

and the coefficients of t^k with k odd are equal to zero since X has property (S).

In our setting X is a finite dimensional variety over $\mathbb{C}$ and we denote by $b_k := b_k(X)$ the number of cells in X with complex dimension k. We set $q := t^2$, $d := \dim X$ and call

$$p_X(q) := \sum_{k=0}^{d} b_k q^k$$

the Poincaré polynomial of the variety X. Sometimes we also refer to b_k as the k-th Betti number. The Euler Poincaré characteristic of X is given by $\chi(X) = p_X(1)$.

In Section 4.4 we define a $\mathbb{C}^*$-action on certain quiver Grassmannians for the equioriented cycle and show that it has finitely many fixed points and that the attracting sets are affine spaces. This implies by the above corollary that these quiver Grassmannians have property (C) and (S). Moreover by a result of G. Cerulli Irelli the Euler-Poincaré characteristic of these quiver Grassmannians equals their number of torus fixed points [**16**, Theorem 1]. Computing the dimension of the attracting sets of the fixed points we obtain the Poincaré polynomial of these quiver Grassmannians.

4.3. $\mathbb{C}^*$-Action on Quiver Grassmannians

In the rest of this chapter we use torus fixed points to compute the Euler-Poincaré characteristic, the Poincaré polynomial and a cellular decomposition of quiver Grassmannians for Δ_n. First we recall two important results by G. Cerulli Irelli which are the foundation for these computations. For certain quiver representations the subsequent theorem shows that the quiver Grassmannians corresponding to these representations have non-negative Euler-Poincaré characteristic.

Theorem 4.7 ([**16**] Theorem 1). *Let M be a Q-representation, $\mathbf{m} := \mathbf{dim}\, M$ its dimension vector and $\chi_{\mathbf{e}}(M)$ the Euler-Poincaré characteristic of the quiver Grassmannian $\mathrm{Gr}^Q_{\mathbf{e}}(M)$. For every $i \in Q_0$ let $\mathcal{B}_i$ be a linear basis of $\Bbbk^{m_i}$ such that for every arrow $\alpha : i \to j$ of Q and every element $b \in \mathcal{B}_i$ there exists an element $b' \in \mathcal{B}_j$ and $\lambda \in \Bbbk$ (possibly zero) such that*

$$M_\alpha b = \lambda b'.$$

Suppose that each $v \in \mathcal{B}_i$ and all its multiples λv for $\lambda \in \Bbbk^*$ is assigned a degree $d(\lambda v) = d(v) \in \mathbb{Z}$ such that:

(D1) for all $i \in Q_0$ all vectors from $\mathcal{B}_i$ have different degrees;

(D2) for every arrow $\alpha : i \to j$ of Q, whenever $b_1 \neq b_2$ are elements of $\mathcal{B}_i$ such that $M_\alpha b_1$ and $M_\alpha b_2$ are non-zero we have:

$$d(M_\alpha b_1) - d(M_\alpha b_2) = d(b_1) - d(b_2).$$

Then

$$\chi_{\mathbf{e}}(M) = \left|\left\{N \in \mathrm{Gr}_{\mathbf{e}}^Q(M) : N(i) \text{ is spanned by a part of } \mathcal{B}_i\right\}\right|.$$

Subrepresentations with the properties as assumed for the set in the last line are called coordinate subrepresentations since their vector spaces are spanned by subbasis of the basis of the surrounding representation. The proof of this theorem is based on the fact that if there is a $\mathbb{C}^*$-action on a complex projective variety with finitely many fixed points, the Euler-Poincaré characteristic of the variety equals the number of fixed points. Constructing a $\mathbb{C}^*$-action on the quiver Grassmannian which has the coordinate subrepresentations as fixed points yields the statement of the theorem.

Let M be a representation in $\mathrm{R}_{\mathbf{m}}(Q)$ and $\mathcal{B}_\bullet$ a collection of basis $\mathcal{B}_i$ of the vector spaces $\Bbbk^{m_i}$ belonging to the representation M.

Definition 4.8. The **coefficient quiver** $\Gamma(M, \mathcal{B}_\bullet)$ is a quiver whose vertices are identified with the elements of $\mathcal{B}_\bullet$ and the arrows are determined as follows: For every arrow $\alpha : i \to j$ of Q and every element $b \in \mathcal{B}_i$ we expand $M_\alpha b$ in the basis $\mathcal{B}_j$ and draw an arrow from $b \in \mathcal{B}_i$ to $b' \in \mathcal{B}_j$ if the coefficient of b' in this expansion is non-zero.

By $T \overrightarrow{\subset} \Gamma(M, \mathcal{B}_\bullet)$ we denote a **successor closed** subquiver T of $\Gamma(M, \mathcal{B}_\bullet)$, i.e. a subquiver T where $\alpha : i \to j$ is an arrow of T when ever it is an arrow of Q and $i \in T_0$ is a vertex of T.

Proposition 4.9 ([**16**] Proposition 1). Let M be a Q-representation satisfying the hypotheses of Theorem 4.7. Then

$$\chi_{\mathbf{e}}(M) = \left|\left\{T \overrightarrow{\subset} \Gamma(M, \mathcal{B}_\bullet) : |T_0 \cap \mathcal{B}_i| = e_i, \text{ for all } i \in Q_0\right\}\right|$$

where T_0 denotes the vertices of T. In particular $\chi_{\mathbf{e}}(M)$ is positive.

Once we established a torus action suiting the requirements of Theorem 4.7, we can use this proposition to determine the Euler-Poincaré characteristic of quiver Grassmannians for Δ_n combinatorially. We have a closer look at coefficient quivers and some examples in Section 4.5.

4.4. Cellular Decomposition

In this section we want to define a torus action on the quiver Grassmannians for Δ_n satisfying the conditions of Theorem 4.7. For this purpose we need the realisation of the quiver Grassmannian coming from the universal Grassmannian. In the previous chapters we worked over an algebraically closed field of characteristic zero. From now on we restrict us to the case $\Bbbk = \mathbb{C}$ because this is required for the computation of the cellular decomposition and some of the results in later chapters are based on the existence of the cellular decomposition. We consider quiver Grassmannians for the representation

$$M := U(\mathbf{d}) := \bigoplus_{i \in \mathbb{Z}_n} U_i(N) \otimes \mathbb{C}^{d_i}$$

and denote its dimension vector by $\mathbf{m} := \mathbf{dim}\, M$. This is the same class of quiver representations as introduced in Chapter 3 but for this section it is more convenient if we do not distinguish between projective and injective representations. A subrepresentation of M is viewed as a collection of vector spaces

$$V := \big(V_i\big)_{i \in \mathbb{Z}_n} \in \prod_{i \in Q_0} \mathrm{Gr}_{e_i}(\mathbb{C}^{m_i}) = \mathrm{Gr}_{\mathbf{e}}(\mathbf{m})$$

which are compatible with the maps M_α corresponding to the quiver representation M, i.e

$$M_i(V_i) \subseteq V_{i+1} \text{ for all } i \in \mathbb{Z}_n.$$

Following Theorem 4.7, we can assume by rescaling that there is a basis

$$\mathcal{B}_i = \left\{ v_k^{(i)} \right\}_{k \in [m_i]}$$

of the space $\mathbb{C}^{m_i}$ for all $i \in \mathbb{Z}_n$ such that $M_i v_k^{(i)}$ is equal to zero or given by a basis vector $v_\ell^{(i+1)} \in \mathcal{B}_{i+1}$. This basis $\mathcal{B}_\bullet$ is called **standard basis** of M. Now we renumber the basis elements such that

$$M_i v_k^{(i)} = v_{k+d_{i+1}}^{(i+1)}$$

holds for all $i \in \mathbb{Z}_n$ whenever it is non-zero. Here d_i is the multiplicity of $U_i(N)$ as summand of M. This order is well defined and unique up to changing the order of the copies of $U_i(N)$. It induces the grading

$$d\big(v_k^{(i)}\big) := k$$

which satisfies the conditions of Theorem 4.7.

A **segment** of M is a maximal collection of vectors $\{v_k^{(i)}\} \subseteq \mathcal{B}_\bullet$ such that there is a unique starting point $v_{k_0}^{(i)}$ and every other element $v_{k'}^{(j)}$ of the collection can be computed by applying a sequence of maps M_α to the starting point which corresponds to a path starting in i and ending in j (possibly going around the cycle more than once). The segments of M correspond to the indecomposable representations $U_i(N)$.

Using this grading, we can define a torus action on the quiver Grassmannian where an element λ of the torus $T := \mathbb{C}^*$ acts on M as

$$\lambda.b := \lambda^{d(b)} b$$

for every element $b \in \mathcal{B}_\bullet$. By linearity this action extends to all elements of M and also to the quiver Grassmannian [**16**, Lemma 1.1].

For a torus fixed point $L \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M)^T$ its **attracting set** is defined as

$$\mathcal{C}(L) := \left\{ V \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M) : \lim_{\lambda \to 0} \lambda.V = L \right\}$$

Following the approach in Section 6.4 of [**19**] we obtain the subsequent result.

THEOREM 4.10. *For every $L \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M)^T$, the subset $\mathcal{C}(L) \subseteq \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M)$ is an affine space and the quiver Grassmannian admits a cellular decomposition*

$$\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M) = \coprod_{L \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M)^T} \mathcal{C}(L).$$

PROOF. We use the action of the torus T on the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M)$ introduced above to define a cell decomposition of the product of classical Grassmannians $\mathrm{Gr}_{\mathbf{e}}(\mathbf{m})$. Considering the torus fixed point L as a collection of subspaces $L \in \mathrm{Gr}_{\mathbf{e}}(\mathbf{m})$ we define the cells

$$\mathcal{C}(L_i) := \Big\{ V \in \mathrm{Gr}_{e_i}(m_i) : \lim_{\lambda \to 0} \lambda.V_i = L_i \Big\}$$

for every $i \in \mathbb{Z}_n$. This induces a decomposition of the classical Grassmannians

$$\mathrm{Gr}_{e_i}(m_i) = \coprod_{L \in \mathrm{Gr}_{e_i}(m_i)^T} \mathcal{C}(L_i).$$

For a representation N in an attracting set of a torus fixed point L we have

$$\begin{aligned} V \in \mathcal{C}(L) &\Leftrightarrow V \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M) \text{ and } \lim_{\lambda \to 0} \lambda.V = L \\ &\Leftrightarrow V \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M) \text{ and } \lim_{\lambda \to 0} \lambda.V_i = L_i \text{ for all } i \in \mathbb{Z}_n \\ &\Leftrightarrow V \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M) \cap \prod_{i \in \mathbb{Z}_n} \mathcal{C}(L_i). \end{aligned}$$

Hence the cell decomposition of the classical Grassmannians is compatible with the structure of the quiver representations, i.e.

$$\mathcal{C}(L) = \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M) \cap \prod_{i \in \mathbb{Z}_n} \mathcal{C}(L_i)$$

and we have the desired cell decomposition of the quiver Grassmannian. It remains to show that the cells in the quiver Grassmannian are affine spaces.

First we show that the cells $\mathcal{C}(L_i)$ are affine spaces. The space $L_i \in \mathrm{Gr}_{e_i}(m_i)$ is spanned by the vectors

$$\Big\{ v_{k_1}^{(i)}, v_{k_2}^{(i)}, \dots, v_{k_{e_i}}^{(i)} \Big\}$$

for some index set $K_i := \{k_1 < k_2 < \dots < k_{e_i}\}$. Here e_i is the i-th entry of the dimension vector $\mathbf{e}$ of the subrepresentations in the quiver Grassmannian. Thus a point $V_i \in \mathcal{C}(L_i)$ is spanned by vectors

$$\Big\{ w_1^{(i)}, w_2^{(i)}, \dots, w_{e_i}^{(i)} \Big\}$$

of the form

$$w_s^{(i)} = v_{k_s}^{(i)} + \sum_{j > k_s, j \notin K_i} \mu_{j,s}^{(i)} v_j^{(i)}$$

with coefficients $\mu_{j,s}^{(i)} \in \mathbb{C}$ because these vectors parametrise all spaces V_i with limit L_i. Hence the cells $\mathcal{C}(L_i)$ are affine spaces for all $i \in \mathbb{Z}_n$.

In order to prove that the cells in the quiver Grassmannian are affine we have to describe the coordinates in the intersection of the cells in the classical Grassmannians with the quiver Grassmannian. Let $V \in \mathcal{C}(L)$ be a point in the some cell. Like above it corresponds to a collection of spaces V with $V_i \in \mathcal{C}(L_i)$ which is parametrised by the collection of coefficients $\{\mu^{(i)}_{j,s}\}$. The conditions $M_\alpha V_{s_\alpha} \subseteq V_{t_\alpha}$ for all arrows α of Δ_n translate to $M_i w^{(i)}_s$ being included in the span of

$$\left\{w^{(i+1)}_1, w^{(i+1)}_2, \dots, w^{(i+1)}_{e_{i+1}}\right\}.$$

If $M_i w^{(i)}_s$ is non-zero, this leads to the equation

$$\begin{aligned} M_i w^{(i)}_s &= M_i v^{(i)}_{k_s} + \sum_{j>k_s, j\notin K_i} \mu^{(i)}_{j,s} M_i v^{(i)}_j \\ &= v^{(i+1)}_{k_s+d_{i+1}} + \sum_{j>k_s, j\notin K_i, M_i v^{(i)}_j \neq 0} \mu^{(i)}_{j,s} v^{(i+1)}_{j+d_{i+1}} \end{aligned}$$

where $v^{(i+1)}_{k_s+d_{i+1}} \in L_{i+1}$ because L is a subrepresentation of M.

This vector is included in V_{i+1} if it lives in the span of

$$\left\{w^{(i+1)}_1, w^{(i+1)}_2, \dots, w^{(i+1)}_{e_{i+1}}\right\}$$

and this is satisfied if and only if

$$\begin{aligned} M_i w^{(i)}_s = w^{(i+1)}_{s+d_{i+1}} &= v^{(i+1)}_{k_s+d_{i+1}} + \sum_{j>k_s+d_{i+1}, j\notin K_{i+1}} \mu^{(i+1)}_{j,s} v^{(i+1)}_j \\ &= v^{(i+1)}_{k_s+d_{i+1}} + \sum_{j>k_s, j+d_{i+1}\notin K_{i+1}} \mu^{(i+1)}_{j+d_{i+1},s} v^{(i+1)}_{j+d_{i+1}} \end{aligned}$$

which leads to the subsequent equality of coefficients

$$\mu^{(i+1)}_{j+d_{i+1},s} = \mu^{(i)}_{j,s} \text{ whenever } M_i v^{(i)}_j \neq 0$$

showing that the cells in the quiver Grassmannian are affine spaces. □

Corollary 4.11. The cellular decomposition of the quiver Grassmannian introduces a cellular decomposition of the strata

$$\mathcal{S}_N = \coprod_{L\in \mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(M)^T : L\cong N} \mathcal{C}(L).$$

Proof. The stratum of N contains all $U \in \mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(M)$ which are isomorphic to N. Thus we have to show that every point $U \in \mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(M)$ is isomorphic to the torus fixed point L attracting it. We show that up to choice of basis U and L have the same image in $\mathrm{R}^{(\mathbf{d})}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$. For this we need the parametrisation of subrepresentations via tuples of subspaces in the vectorspaces over the vertices of Δ_n.

Let

$$\left\{v^{(i)}_{k_1}, v^{(i)}_{k_2}, \dots, v^{(i)}_{k_{e_i}}\right\}$$

be the basis for the vectorspaces over the vertices of Δ_n for the torus fixed point L which is obtained as a subbasis of the basis for the vectorspaces corresponding to the representation M.

Isomorphism classes of representations in $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ and orbits of the group $\mathrm{GL}_{\mathbf{e}}$ in $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ coincide. For an element U of the attracting set $\mathcal{C}(L)$ let

$$\left\{w^{(i)}_1, w^{(i)}_2, \dots, w^{(i)}_{e_i}\right\}$$

be the basis for its vectorspaces over the vertices. By Theorem 4.10 we know that the coefficients expressing this basis in the basis of L are subject to the conditions

$$\mu^{(i+1)}_{j+d_{i+1},s} = \mu^{(i)}_{j,s} \text{ whenever } M_i v^{(i)}_j \neq 0.$$

These equations together with the parametrisation

$$w^{(i)}_s = v^{(i)}_{k_s} + \sum_{j>k_s, j\notin K_i} \mu^{(i)}_{j,s} v^{(i)}_j$$

yield that the maps M_i act on both basis in the same way, i.e.

$$M_i v^{(i)}_{k^{(i)}_j} = v^{(i+1)}_{k^{(i)}_j+1} \text{if and only if } M_i w^{(i)}_j = w^{(i+1)}_{j+1}.$$

Accordingly the restrictions of the maps M_i for $i \in \mathbb{Z}_n$ to the basis of the subspaces coincide for the representations U and L. Hence both subrepresentations are isomorphic and the strata decompose into the attracting sets of torus fixed points. □

Corollary 4.12. The possible types of subrepresentations in $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(M)$ are given by the fixed points of the torus action.

Proof. The subrepresentation types are given by the strata. In Section 3.5.2 we computed that the strata in the quiver Grassmannian are parametrised as

$$\mathcal{S}^{(\mathbf{d})}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N) = \Big\{ \mathbf{l} := (\ell_{i,k}) \in \bigoplus_{i\in\mathbb{Z}_n} \bigoplus_{k=1}^{d_i} [N]_0 : \ \mathbf{dim}\, U(\mathbf{l}) = \mathbf{e} \text{ and } \ell_{i,k} \geq \ell_{i,k+1} \Big\}.$$

From this information we can directly construct one successor closed subquiver in the the coefficient quiver of M which has this subrepresentation type. The other way around we take the length of the segments of a successor closed subquiver and rearrange them such that they satisfy $\ell_{i,k} \geq \ell_{i,k+1}$ for all $i \in \mathbb{Z}_n$ and all $k \in [d_i - 1]$. The resulting tuple is contained in the set above and parametrises a subrepresentation type of M. □

This corollary will be helpful to deduce information about the structure of the stratification from the torus fixed points as done in Section 4.7. Moreover it allows us to prove Proposition 3.19.

Proof of Proposition 3.19. The subrepresentations of a quiver representation M are determined by the successor closed subquivers in the coefficient quiver of M. There exists an embedding $U_i(\ell) \hookrightarrow U_j(k)$ if and only if $i+\ell = j+k$ and $k \geq \ell$. This is equivalent to the segment of $U_i(\ell)$ being a successor closed subsegment of the segment corresponding to $U_j(k)$. Thus the embedding of a representation into M can be decomposed into the embeddings of the subsegments which are of the form

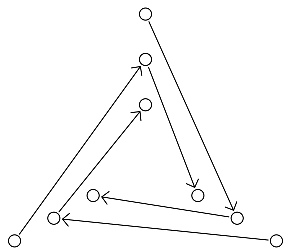
$$U_i(\ell) \hookrightarrow U_{i-k}(\ell + k).$$

□

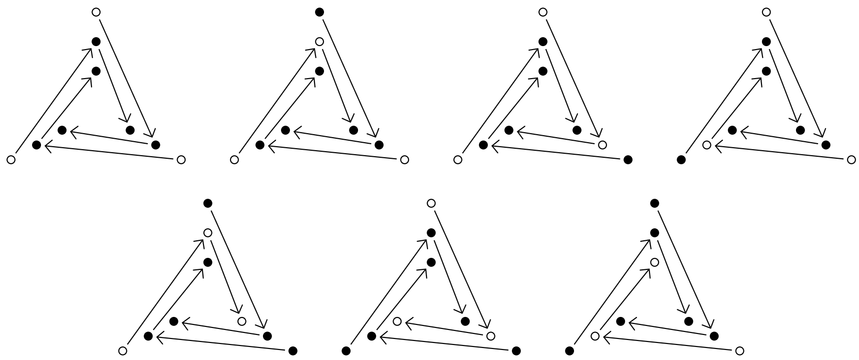
4.5. Coefficient Quivers

In this subsection, we want to give some examples of coefficient quivers and their use before we turn our attention to formulas for the Euler characteristics and the Poincaré polynomials.

EXAMPLE 4.13. Let $n = N = 3$ and define $X := U_1(3) \oplus U_2(3)$ and $Y := U_3(3)$. We compute the dimension of $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ and its Poincaré polynomial using the coefficient quiver. Following the order of the vector space basis we defined above, the coefficient quiver of $X \oplus Y$ is given by



In this coefficient quiver we mark the subquivers using black vertices. The following list contains all successor closed subquivers T of this quiver satisfying $|T_0 \cap \mathcal{B}_i| = e_i = 2$ for all $i \in \mathbb{Z}_3$.



The dimensions of the cells could be directly read of from the picture. We have to count the white vertices between each starting point of a segment and the center of the picture. For simplicity we will refer to this as below a point. The dimension of the cell is given by the sum of these numbers.

Accordingly the first cell is zero-dimensional. The next three cells are one-dimensional since there is only one white vertex below one of the three starting points of the segments in each picture. For the last three pictures we have one white vertex below two of the three starting points and the dimension of these cells sums up to two.

Collecting this information the Poincaré polynomial of $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(X \oplus Y)$ is given by

$$p_{\mathbf{e},X\oplus Y}(t) = 3t^2 + 3t + 1.$$

Hence the dimension of the Grassmannian is 2 and its Euler characteristic is 7.

REMARK. The approach to compute the cell dimension as described above works in the full generality of Theorem 4.10.

PROOF. The points in the cells are described by the coefficients

$$\Big\{\mu^{(i)}_{j,s} : i \in \mathbb{Z}_n, s \in [e_i], j > k_s \text{ and } j \notin K_i\Big\}$$

which are subject to the relations

$$\mu^{(i+1)}_{j+d_{i+1},s} = \mu^{(i)}_{j,s} \text{ whenever } M_i v^{(i)}_j \neq 0.$$

The number of free parameters in this set is equal to the dimension of the corresponding cell. By the order of the vector space basis it is clear that the number of parameters $\mu^{(i)}_{j,s}$ in the set above is the biggest for the vector corresponding to the starting point of a segment.

The parameters for the later points of a segment are all determined by these starting parameters since they have to satisfy the relations above. Hence the number of free parameters is given as the sum of parameters for the vectors corresponding to the starting points of the segments. By the restrictions $j > k_s$ and $j \notin K_i$ these numbers are given by the number of holes below the starting points of the the segments. □

4.6. Euler Characteristics and Poincaré Polynomials

We use the $\mathbb{C}^*$-action defined above to compute the Euler characteristic and the Poincaré polynomial of quiver Grassmannians for the equioriented cycle.

REMARK. The parametrisation of the strata in the quiver Grassmannians as introduced in Section 3.5.2 is also suitable to compute the cell structure of the quiver Grassmannians since cells are in bijection with successor closed subquivers and for the equioriented cycle these subquivers are parametrised by the set

$$\mathcal{C}^{(\mathbf{d})}_{\mathbf{e}}\big(\Delta_n, \mathrm{I}_N\big) := \Big\{\mathbf{l} := (\ell_{i,k}) \in \bigoplus_{i\in\mathbb{Z}_n} [N]_0^{d_i} : \ \mathbf{dim}\, U(\mathbf{l}) = \mathbf{e}\Big\}.$$

Introducing a new dimension function arising from the coefficient quivers we can use this set to compute the Poincaré series of the quiver Grassmannian

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}}\big(J\big)$$

and the cardinality of this set equals the Euler Poincaré characteristic of the quiver Grassmannian. The code for a program to compute Poincaré polynomials for approximations of the affine flag variety and the affine Grassmannian which is based on this parametrisation is presented in Appendix B. Some of the results of the computations are presented in the Appendix C.

Despite this computational description for the Euler characteristic and the Poicaré polynomial we want to have closed formulas for this data. In the rest of this section we examine some cases where it is possible to constract such formulas from the coefficient quiver parametrisation of the cells.

PROPOSITION 4.14. Let $M := \bigoplus_{i\in\mathbb{Z}_n} U_i(n)$, $\mathbf{e} := \mathbf{dim}\, U_i(n)$ and denote the Euler characteristic of the corresponding quiver Grassmannian by $\chi_{\mathbf{e}}(M)$ and let $p_{\mathbf{e},M}(q)$ denote its Poicaré polynomial. Then

(i) $\dim \mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(M) = n - 1$
(ii) $\chi_{\mathbf{e}}(M) = 2^n - 1$
(iii) $p_{\mathbf{e},M}(q) = (q+1)^n - q^n.$

Proof. Part (i): From Lemma 3.22 we know

$$\dim \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M) = k(m-k) = 1(n-1) = n-1$$

where in this setting $k = e_i = 1$ and m is the sum over the multiplicities of the $U_i(n)$ as summand of M which is given by n.

Part (iii): For every $k \in \{0, 1, \ldots, n-1\}$ we have to determine the successor closed subquivers of the coefficient quiver of M corresponding to the cells of dimension k. Since $\mathbf{e} = (1, \ldots, 1)$, we have to partition $\mathbb{Z}_n$ into at most n non-overlapping intervals in order to find a subrepresentation of M with dimension vector $\mathbf{e} = (1, \ldots, 1)$. All $U_i(n)$ occur as summand of M with multiplicity one. Hence the number of cells is equal to the number of subrepresentations and both are in bijection with the partitions of $\mathbb{Z}_n$.

The dimension of a cell depends only on the length of the intervals in the corresponding partition because the segments are non-overlapping and the dimension of a cell could be read of from the subquiver of the coefficient quiver corresponding to the cell by counting the number of free points below the starting points of the segments as shown in Section 4.5.

Namely in this setting the dimension of a cell is given by the sum of the length of a segment over all segments belonging to the cell, where the length of a segment is given by the number of arrows from the coefficient quiver contained in the segment. This number is the same as n minus the number of segments corresponding to the cell.

Accordingly for $k = 0$ we have to partition $\mathbb{Z}_n$ into n segments of length zero and there is only one possibility to do this. For $k = n-1$, we have to take one segment of length n and there are n possibilities to do this.

In general there is a unique partition of $\mathbb{Z}_n$ into ℓ segments for every choice of ℓ starting points of the segments and this gives all partitions into ℓ segments. The dimension of the corresponding cell is $n - \ell$ and there are $\binom{n}{\ell} = \binom{n}{n-\ell}$ possibilities to choose the starting points of the intervals. Summing them up we obtain the formula

$$p_{\mathbf{e},M}(q) = \sum_{k=0}^{n-1} \binom{n}{k} q^k.$$

Finally we show

$$\sum_{k=0}^{n} \binom{n}{k} q^k = (q+1)^n$$

by induction and plug it into the formula.

Part (ii): By Theorem 4.7 the Euler characteristic equals the number of torus fixed points. Hence we get $\chi_{\mathbf{e}}(M) = p_{\mathbf{e},M}(1)$ which leads to

$$\chi_{\mathbf{e}}(M) = p_{\mathbf{e},M}(1) = \sum_{k=0}^{n} \binom{n}{k} 1^k - \binom{n}{n} 1^n = (1+1)^n - 1 = 2^n - 1.$$

□

LEMMA 4.15. *Let* $M := \bigoplus_{i\in\mathbb{Z}_n} U_i(n)\otimes\Bbbk^{d_i}$, $m := \sum_{i\in\mathbb{Z}_n} d_i$, $\mathbf{e} := \mathbf{dim}\, U_i(n)$. *Then*

$$(i) \quad \dim \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M) = \sum_{i\in\mathbb{Z}_n} d_i - 1$$

$$(ii) \quad \chi_{\mathbf{e}}(M) = \prod_{i\in\mathbb{Z}_n}(d_i+1) - 1$$

$$(iii) \quad p_{\mathbf{e},M}(q) = \prod_{i\in\mathbb{Z}_n}\frac{q^{d_i+1}-1}{q-1} - q^m.$$

PROOF. Part (i) follows from Lemma 3.22 again.

Part (ii): We prove this statement by induction over the d_i's. The beginning of the induction is the result from Proposition 4.14. Without loss of generality we assume that we get $\mathbf{d}'$ from $\mathbf{d}$ by increasing the i-th entry by one. Then

$$\chi_{\mathbf{e}}(M') = \chi_{\mathbf{e}}(M) + \big|\{\text{cells using the new } U_i(n)\}\big|$$

where the set above contains alls cells with a segment in the new part of the coefficient quiver. Now we have to determine the cardinality of this set.

For every $I \subseteq \mathbb{Z}_n$ we can define a unique representation with dimension vector $\mathbf{e} = (1,\dots,1)$ corresponding to the set I as we did in the proof of Proposition 4.14. In the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M)$ for

$$M = \bigoplus_{i\in\mathbb{Z}_n} U_i(n)\otimes\Bbbk^{d_i}$$

there are $\prod_{i\in I} d_i$ cells corresponding to this representation.

For a fixed $i\in\mathbb{Z}_n$ with $d_i>0$ we consider $U_{i+n-\ell}(\ell)$ for arbitrary ℓ. There are 2^{n-1} possibilities to find an $I\subseteq\mathbb{Z}_n$ with $i\in I$ and to every I we can assign a unique representation. The number of all cells corresponding to these representations is given by

$$\prod_{j\in\mathbb{Z}_n, j\neq i}(d_j+1).$$

Applying the induction hypothesis we obtain

$$\begin{aligned}\chi_{\mathbf{e}}(M') &= \prod_{j\in\mathbb{Z}_n}(d_j+1) - 1 + \prod_{j\in\mathbb{Z}_n, j\neq i}(d_j+1)\\ &= (d_i+2)\prod_{j\in\mathbb{Z}_n, j\neq i}(d'_j+1) - 1 = \prod_{j\in\mathbb{Z}_n}(d'_j+1) - 1.\end{aligned}$$

Part (iii): In this case the number of free points below the starting points of the segments depends on the length of the segments and the index $k_i\in\{1,\dots,d_i\}$ of the copy of $U_i(n)$ we embed it in. As seen before the length is determined by the starting points and thus the cells are in bijection with tuples of the k_i's. By this correspondence every cell is in bijection with one factor in the product

$$\prod_{i\in\mathbb{Z}_n}\Big(\sum_{k=0}^{d_i} q^k\Big)$$

where we take the power $d_i - k_i$ whenever a segment embedded into the k_i-th copy of $U_i(n)$ belongs to the cell and otherwise the power will be d_i. Then the cell dimension is equal to the exponent of q and in the polynomial its coefficient counts the number of cells with this dimension. Hence the only factor in this product which is not in bijection with the cells is

$$q^{\sum_{i\in\mathbb{Z}_n} d_i}$$

which we have to subtract. The formula follows by writing the sums as fractions, i.e.

$$\sum_{k=0}^{d_i} q^k = \frac{q^{d_i+1}-1}{q-1}.$$

□

Yet we are only able to give closed formulas for the Poincaré polynomial if every entry of the dimension vector $\mathbf{e}$ is equal to one. For other dimension vectors $\mathbf{e}$ we would have to consider partitions of multiples of n if we still assume it to be homogeneous. In the case that $\mathbf{e}$ is not homogeneous it is not possible to control the cells using partitions.

Furthermore this proof relies on the fact that for every i at most one segment is embedded into the copies of $U_i(N)$ which is not true for the entries of the dimension vector $\mathbf{e}$ being larger than one. But in the more general setting there are still some symmetries.

PROPOSITION 4.16. Consider the representation $M := \bigoplus_{i\in\mathbb{Z}_n} U_i(n)\otimes \Bbbk^{d_i}$, the dimension vector of subrepresentations $\mathbf{e} := \mathbf{dim}\, U_i(n)\otimes \Bbbk^q$ and set $\mathbf{m} := \mathbf{dim}\, M$. Then

$$p_{\mathbf{e},M}(q) = p_{\mathbf{m}-\mathbf{e},M}(q).$$

PROOF. The quiver Δ_n is self dual and every cell in the coefficient quiver of M with dimension vector $\mathbf{e}$ corresponds to a cell with dimension vector $\mathbf{m}-\mathbf{e}$ for the dual of M which is isomorphic to M. □

Moreover the Poincaré polynomial can not detect permutations of the d_i's. But this equality does not have to come from an isomorphism of the quiver Grassmannians.

EXAMPLE 4.17. Let $N = n = 4$ and consider the tuples of multiplicities of indecomposable nilpotent representations $\mathbf{d} := {}^t(1,2,1,2)$, $\hat{\mathbf{d}} := {}^t(1,1,2,2)$ and the dimension vector $\mathbf{e} := {}^t(2,2,2,2)$. Then we have an equality of Poincaré polynomials

$$p_{\mathbf{e},U(\mathbf{d})}(q) = p_{\mathbf{e},U(\hat{\mathbf{d}})}(q) = 8q^8 + 24q^7 + 43q^6 + 48q^5 + 40q^4 + 24q^3 + 12q^2 + 4q + 1$$

for the quiver Grassmannians $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}}\big(U(\mathbf{d})\big)$ and $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}}\big(U(\hat{\mathbf{d}})\big)$. The quiver Grassmannian for $\mathbf{d}$ decomposes into 66 strata whereas for $\hat{\mathbf{d}}$ we get a decomposition into 65 strata. Hence the structures of the Grassmannians are different and there is no isomorphism between them.

We are not able to give an explanation in the general setting but can show that two Grassmannians with the same Poincaré polynomial share the same motive using the Cut and Paste property and isomorphisms at the level of cells which are affine spaces by Theorem 4.10. For definitions and more detail on this see the articles by T. Beke [**4**] and M. Larsen and V. Lunts [**57**]. There are even quiver Grassmannians for Δ_n with the same Poincaré polynomial and the same stratification which are still non-isomorphic.

EXAMPLE 4.18. Let $N = n = 4$, $\mathbf{d} := {}^t(1,2,3,2)$, $\hat{\mathbf{d}} := {}^t(1,3,2,2)$ and take the dimension vector $\mathbf{e} := {}^t(2,2,2,2)$. Then

$$p_{\mathbf{e},U(\mathbf{d})}(q) = p_{\mathbf{e},U(\hat{\mathbf{d}})}(t) = 9q^{12} + 31q^{11} + 71q^{10} + 112q^9 + 142q^8 + 143q^7 + 123q^6 + 8q^5 + 56q^4 + 29q^3 + 13q^2 + 4q + 1.$$

Moreover the quiver Grassmannians $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}\big(U(\mathbf{d})\big)$ and $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}\big(U(\hat{\mathbf{d}})\big)$ have the same stratification. But there is no isomorphism between the quiver Grassmannians since it is not possible to match the structure of the stratifications and the cellular decompositions, i.e. there are strata with the same representative but different cellular decompositions in the different quiver Grassmannians.

4.7. Application to Stratification

In this section we use the coefficient quiver combinatorics to deduce information about the stratification of the quiver Grassmannians.

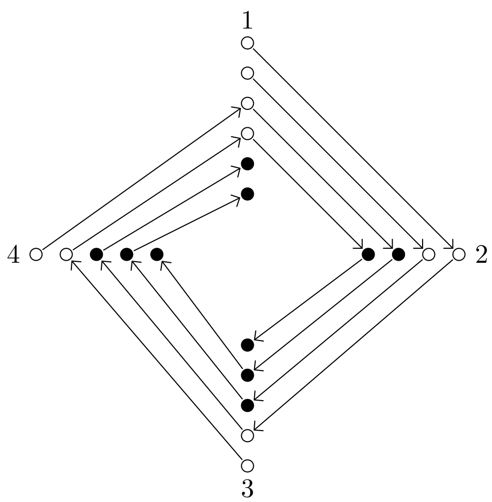
LEMMA 4.19. *In every Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M)$ there is a unique stratum of smallest dimension which is included in the closure of every other stratum. A representative $B_{\mathbf{e}}$ can be found taking the subrepresentation of M corresponding to the e_i inner points of the coefficient quiver at every vertex $i \in \mathbb{Z}_n$.*

In the following, we refer to $B_{\mathbf{e}}$ as **base** of the stratification. The cell of this stratum as obtained in the proof of Lemma 4.19 is zero-dimensional. With the base of the stratification we are able to compute the full stratification of the quiver Grassmannian using the methods concerning degenerations of orbits and singularities as developed in the thesis of G. Kempken and recalled in Section 3.1.

EXAMPLE 4.20. Let $n = 4, N = 5$, $X := U_3(5) \oplus U_4(5)$, $Y := U_1(5) \oplus U_1(5)$ and

$$\mathbf{e} := \mathbf{dim}\, X = {}^t(1,1,2,1) + {}^t(1,1,1,2) = {}^t(2,2,3,3).$$

The successor closed subquiver in the coefficient quiver of $X \oplus Y$ corresponding to the stratum of smallest dimension is given as



This gives the representative

$$B_{\mathbf{e}} = U_2(3) \oplus U_3(3) \oplus U_2(2) \oplus U_4(2)$$

and its stratum is one-dimensional. The dimension of the stratum is obtained as the maximum over the dimension of all cells with this representative.

Proof of Lemma 4.19. Define the tuple $\mathbf{q}$ component wise as

$$q_i := \min\{d_i, e_i\}$$

and consider the representation

$$S_{\mathbf{q}} := \bigoplus_{i \in \mathbb{Z}_n} S_i \otimes \Bbbk^{q_i}.$$

This representation is unique by construction and its orbit $\mathcal{O}_{S_{\mathbf{q}}}$ in the variety of quiver representations is zero-dimensional. If $\mathbf{q} = \mathbf{e}$, this finishes the proof because $S_{\mathbf{q}}$ is contained in the orbit closure of every representation in $\mathrm{R}_{\mathbf{q}}(\Delta_n, \mathrm{I}_N)$ and the end points of the segments in the coefficient quiver of M are the inner points over the vertices.

Now assume that $\mathbf{q} \neq \mathbf{e}$ and that there is at least one index $i \in \mathbb{Z}_n$ such that $q_i < e_i$. Starting at one $i \in \mathbb{Z}_n$ with $q_i < e_i$ and $q_{i+1} > 0$ enlarge the shortest segment of $B := S_{\mathbf{q}}$ ending at $i+1 \in \mathbb{Z}_n$ by one. If there is no unique segment with this property, enlarge the lowest one of the shortest segments in the coefficient quiver.

We order the elements $j \in \mathbb{Z}_n$ by the length the shortest path from j to i. The largest element in this order is i and the second largest is $i-1$. We do the same enlargement as above for the next biggest $j < i$ with $q_j < e_j$. Here $q_{j+1} > 0$ is not required since we take the largest j. We repeat this procedure until $\mathbf{dim}\, B = \mathbf{e}$. It is clear that this method terminates in a subrepresentation of M with the desired dimension vector but it might be necessary to go around the circle more than once until there are no more $i \in \mathbb{Z}_n$ with $q_i < e_i$.

This algorithm gives the same representative $B_{\mathbf{e}}$ as the approach using the coefficient quiver because each step in the algorithm corresponds to taking the next lowest point in the coefficient quiver of M.

By construction, the representation $B_{\mathbf{e}}$ is unique up to permutation of the summands and the words corresponding to the summands are as short as possible for a subrepresentation of M with dimension vector $\mathbf{e}$. The orbit $\mathcal{O}_{B_{\mathbf{e}}}$ is included in the orbit closure inside $\mathrm{R}_{\mathbf{e}}(\Delta_n, \mathrm{I}_N)$ for any $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M)$ by Proposition 3.10. Hence its dimension has to be minimal among all elements of the quiver Grassmannian and the same holds for the stratification of the Grassmannian because of Theorem 2.3. □

Let $\mathbf{y}$ be a n-tuple of multiplicities of injective nilpotent representations I_j and let $\mathbf{x}$ be a tuple of multiplicities of projective nilpotent representations P_i. By Corollary 3.3 we obtain

$$P_i \cong U_i(N) \quad \text{and} \quad I_j \cong U_{j-N+1}(N)$$

for $P_i, I_j \in \mathrm{rep}_{\Bbbk}(\Delta_n, \mathrm{I}_N)$. We define

$$I_{\mathbf{d}} := \bigoplus_{i \in \mathbb{Z}_n} U_{i-N+1}(N) \otimes \Bbbk^{d_i}$$

where $d_i := y_i + x_{i-N+1}$ for all $i \in \mathbb{Z}_n$. For the tuple $\mathbf{x}$ we set

$$P_{\mathbf{x}} := \bigoplus_{i\in\mathbb{Z}_n} U_i(N) \otimes \Bbbk^{x_i}.$$

The tuple of multiplicities $\mathbf{q}$ is defined entry wise as

$$q_i := \min\{d_i, e_i\}$$

where e_i is the i-th entry of the dimension vector $\mathbf{e} := \mathbf{dim}\, P_{\mathbf{x}}$. Observe that $S_i := U_i(1) \hookrightarrow U_{i-N+1}(N)$ and hence $S_{\mathbf{e}} \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(I_{\mathbf{e}})$ where

$$S_{\mathbf{e}} := \bigoplus_{i\in\mathbb{Z}_n} S_i \otimes \Bbbk^{e_i}.$$

LEMMA 4.21. *The images of the projections from the quiver Grassmannians* $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}\big(I_{\mathbf{d}}\big)$ *and* $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(I_{\mathbf{q}})$ *to the variety of quiver representations coincide, i.e.*

$$\mathrm{R}_{\mathbf{e}}^{(\mathbf{d})}\big(\Delta_n, \mathrm{I}_N\big) \cong \mathrm{R}_{\mathbf{e}}^{(\mathbf{q})}\big(\Delta_n, \mathrm{I}_N\big).$$

PROOF. The types of subrepresentations obtained from torus fixed points are the same in both quiver Grassmannians. □

REMARK. This Lemma proves that the two quiver Grassmannians in Example 4.18 have the same stratification.

Hence for a fixed dimension vector $\mathbf{e}$ the stratification of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}\big(I_{\mathbf{d}}\big)$ stays the same if we increase entries of $\mathbf{d}$ which are already bigger than the corresponding entry of $\mathbf{e}$ whereas the number of torus fixed points of this Grassmannian grows exponentially fast. Moreover for a fixed $\mathbf{e}$ there are only finitely many $\mathbf{d}$'s leading to different stratifications of $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}\big(I_{\mathbf{d}}\big)$.

It is possible to classify these $\mathbf{d}$'s using the property above and the fact that the structure of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}\big(I_{\mathbf{d}}\big)$ does not change with cyclic permutations of the entries in $\mathbf{e}$ and $\mathbf{d}$.

CHAPTER 5

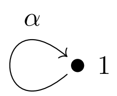
The Affine Grassmannian and the Loop Quiver

The loop quiver is an equioriented cycle for $n = 1$. In this chapter we want to apply the theory developed for the quiver Grassmannians for the equioriented cycle in Chapter 3 to study finite approximations of the affine Grassmannian. We define linear degenerations of the affine Grassmannian and study the finite approximations of the partial degenerations based on the same approach. For the Feigin degeneration of the affine Grassmannian, these approximations were developed and studied by E. Feigin, M. Finkelberg and M. Reineke in [**30**].

The constructions and methods used in this chapter are similar to what is needed for the study of the affine flag variety in Chapter 6. Both chapters are independent of each other but in this chapter the notation is less complicated. Hence it can be viewed as motivation or preparation to the more general constructions given in the chapter about the affine flag variety.

5.1. The Loop Quiver

The loop is a special case of an equioriented cycle and some of the results about quiver Grassmannians for the cycle can be sharpened or simplified in notation for the loop quiver. In this section we modify the constructions for the equioriented cycle Δ_n as introduced in Chapter 3 to the special case of the loop quiver

$\Delta_1 =$ 

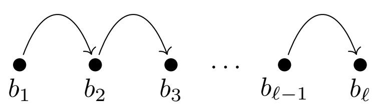
The loop has only one arrow such that all paths consist of concatenations of this arrow. Hence the path algebra $\mathbb{C}\Delta_1$ is isomorphic to the polynomial ring in one variable $\mathbb{C}[t]$. The generating paths of the ideal $\mathrm{I_N}$ boil down to the single path

$$p(N) := (\,1 \,|\, \alpha^N \,|\, 1\,).$$

Accordingly the bounded path algebra is isomorphic to the truncated polynomial ring, i.e.

$$A_N := \mathbb{C}\Delta_1/\mathrm{I}_N \cong \mathbb{C}[t]/(t^N).$$

The indecomposable nilpotent representations as determined by Proposition 3.2 are denoted by U_ℓ and have coefficient quivers of the shape



They are all isomorphic to a truncated polynomial ring, i.e.

$$U_\ell \cong \mathbb{C}[t]/(t^\ell) \cong A_\ell.$$

Moreover the projective and injective representations in $\mathrm{rep}(\Delta_1, \mathrm{I}_N)$ are isomorphic and there is exactly one indecomposable projective/injective bounded representation since the loop has only one vertex, i.e. $P_1^{(N)} \cong I_1^{(N)}$.

For the loop quiver, the class of quiver Grassmannians we introduced in Chapter 3 for the equioriented cycle reads as

$$\mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^{x+y}\big)$$

where $x, y \in \mathbb{N}$. It is possible to apply the results from Chapter 2 since A_N is an injective representation of the loop quiver.

5.1.1. Homomorphisms of Representations of the Loop Quiver and Words. The alphabet (i.e. the set of vertices) for the words corresponding to indecomposable representations of the loop quiver only consist of one letter. Hence the dimension of the space of homomorphisms between indecomposable representations of the loop is given by the length of the shorter word.

PROPOSITION 5.1. For two indecomposable representations U_ℓ and U_k of the loop quiver, the dimension of the space of morphisms from U_ℓ to U_k is given by

$$\dim \mathrm{Hom}_{\Delta_1}\big(U_\ell, U_k\big) = \min\big\{\ell, k\big\}.$$

PROOF. The loop quiver is an equioriented cycle with only one vertex. The words corresponding to the indecomposable representation are repetitions of the same letter. Adapting Proposition 3.16 to this setting yields the claimed formula. □

This statement can also be proven by a direct computation using the shape of the maps belonging to the quiver representations U_ℓ and U_k and the commutativity relations defining morphisms of quiver representations.

5.1.2. Geometry of Quiver Grassmannians for the Loop Quiver. The formula for the dimension of the space of homomorphisms allows us to compute the dimension of this special class of quiver Grassmannians.

PROPOSITION 5.2. Let $N \in \mathbb{N}$ and $x, y \in \mathbb{N}$. Then the dimension of the quiver Grassmannian for the loop quiver computes as

$$\dim \mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^{x+y}\big) = Nxy.$$

PROOF. Since the loop quiver has only one vertex, we can apply Proposition 3.21 to every N and obtain

$$\dim \mathrm{Hom}_{\Delta_1}\big(U, U\big) \geq \dim \mathrm{Hom}_{\Delta_1}\big(A_N \otimes \mathbb{C}^x, A_N \otimes \mathbb{C}^x\big)$$

for all $U \in \mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^{x+y}\big)$. Accordingly the dimension of the quiver Grassmannian computes as

$$\begin{aligned}
&\dim \mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^{x+y}\big)\\
&\quad= \dim \mathrm{Hom}_{\Delta_1}\big(A_N \otimes \mathbb{C}^x, A_N \otimes \mathbb{C}^{x+y}\big) - \dim \mathrm{Hom}_{\Delta_1}\big(A_N \otimes \mathbb{C}^x, A_N \otimes \mathbb{C}^x\big)\\
&\quad= \dim \mathrm{Hom}_{\Delta_1}\big(A_N \otimes \mathbb{C}^x, A_N \otimes \mathbb{C}^y\big)\\
&\quad= \dim \mathrm{Hom}_{\Delta_1}\big(A_N, A_N\big)xy\\
&\quad= Nxy.
\end{aligned}$$

Here the last equality follows by Proposition 5.1 and the linearity of the space of homomorphisms yields the other equalities. □

Proposition 5.3. The quiver Grassmannian

$$\mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^{x+y}\big)$$

is irreducible, normal, Cohen-Macaulay and has rational singularities.

Proof. By Lemma 3.24, we know that the irreducible components of this quiver Grassmannian satisfy these properties because the loop quiver is an oriented cycle with one vertex. The parametrising set of the irreducible components as determined in Lemma 3.23 is given as

$$C_x(x+y) := \Big\{ p \in \mathbb{Z}_{\geq 0} : p \leq x+y, p = x \Big\}.$$

It contains only the element $p = x$. □

5.1.3. Parametrisation of the Image in the Variety of Quiver Representations and the Orbits therein. The parametrisation of the image of the quiver Grassmannian for the loop quiver in the variety of quiver representations by relations and morphisms is given in the subsequent proposition.

Proposition 5.4. For $m := x + y$, the image of the quiver Grassmannian

$$\mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^{m}\big)$$

in the variety of quiver representations is parametrised as

$$\mathrm{R}_{xN}^{(m)}(\Delta_1, \mathrm{I}_N) = \Big\{ U \in \mathrm{R}_{xN}(\Delta_1) : \ U_\alpha^N \equiv 0 \text{ and } \operatorname{corank} U_\alpha \leq m \Big\}.$$

Proof. For the loop quiver there is only one generating relation of I_N because there is only one arrow. The computations in Section 3.5.1 concerning the dimension of the space of morphisms are true for an arbitrary number of vertices in the cycle. Hence we obtain a similar description of representations embedding into $A_N \otimes \mathbb{C}^m$ based on the rank of the map U_α. □

For the parametrisation of the orbits based on the decomposition into indecomposable representations we obtain the following result.

Proposition 5.5. The GL_{xN}-orbits in the variety of quiver representations $\mathrm{R}_{xN}(\Delta_1, \mathrm{I}_N)$ which correspond to strata of the quiver Grassmannian

$$\mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^{m}\big)$$

are parametrised by the set

$$\mathcal{S}_{xN}^{(m)}\big(\Delta_1, \mathrm{I}_N\big) := \Big\{ \mathbf{l} \in [N]_0^m : \ \sum_{k=1}^{m} \ell_k = xN \text{ and } \ell_k \geq \ell_{k+1} \Big\}.$$

Proof. The dimension vectors for representations of the loop quiver have only one entry. We obtain that the dimension vector of a indecomposable representation is equal to the length of the word corresponding to the representation. Hence the condition about the dimension vector of the subrepresentation can be replaced by a summation of the length of its direct summands. The other simplifications also come with setting $n = 1$. □

5.1.4. $\mathbb{C}^*$-Action and Cellular Decomposition of Quiver Grassmannians for the Loop Quiver. Define $M_N := A_N \otimes \mathbb{C}^{x+y}$. The vector space V of the quiver representation M_N over the single vertex of the loop quiver has dimension mN where $m := x + y$. We label the standard basis of the vector space V by

$$\mathcal{B} := \{v_1, v_2, \dots v_{mN-1}, v_{mN}\}.$$

The grading

$$d(v_k) := k$$

of the basis $\mathcal{B}$ satisfies the assumptions of Theorem 4.7. Hence the Euler Poincaré characteristic of the Grassmannian $\mathrm{Gr}_{xN}^{A_N}(M_N)$ is given by the number of torus fixed points in this Grassmannian for the action

$$\mathbb{C}^* \times V \to V; \quad (\lambda, b) \mapsto \lambda.b := \lambda^{d(b)}b.$$

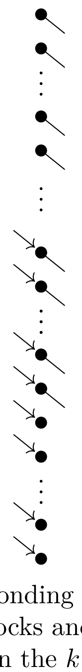
Restricting the number of vertices to one we derive the subsequent statement from Theorem 4.10. The proof of the special case works analogous to the general version.

Proposition 5.6. For every torus fixed point $L \in \mathrm{Gr}_{xN}^{A_N}(M_N)^T$, the attracting set $\mathcal{C}(L) \subseteq \mathrm{Gr}_{xN}^{A_N}(M_N)$ is an affine space and the quiver Grassmannian admits a cellular decomposition

$$\mathrm{Gr}_{xN}^{A_N}(M_N) = \coprod_{L \in \mathrm{Gr}_{xN}^{A_N}(M_N)^T} \mathcal{C}(L).$$

By Proposition 4.9 we know that we can count successor closed subquivers in the coefficient quiver of M_N in order to determine the number of torus fixed points of the quiver Grassmannian.

The coefficient quiver of the representation M_N can be drawn as follows



In each block we have m dots corresponding to the m indecomposable representations M_N consists of. There are N blocks and the i-th point in the k-th block has one outgoing arrow to the i-th point in the $k+1$-th block. For better visibility we decide to draw no complete arrows. Otherwise we would have very long or a lot of crossing arrows in the planar picture.

The successor closed subquivers described by Proposition 4.9 contain xN of the mN points in the coefficient quiver of M_N. Each subsegment in the m segments has to be successor closed. Hence it is uniquely described by its length which can vary between zero and N. Collection the length of all subsegments in the segments of M_N is sufficient to describe the corresponding successor closed subquiver. This yields the subsequent parametrisation of the cells.

Proposition 5.7. The cells of the quiver Grassmannians for the loop are parametrised by the set

$$\mathcal{C}_{xN}^{(m)}\big(\Delta_1, \mathrm{I}_N\big) := \Big\{\mathbf{l} \in [N]_0^m : \sum_{k=1}^{m} \ell_k = xN\Big\}.$$

The cardinality of this set is equal to the Euler Poincaré characteristic of these Grassmannians. In Section 5.8 we compute the cardinality of this set for the special case $x = y$. We can also use this set to compute the Poincaré polynomial for the quiver Grassmannian by defining a function computing the dimension of the cells which is based on this parametrisation.

5.1.5. Poincaré polynomials of Quiver Grassmannians for the Loop Quiver. This section is devoted to the description of the Poincaré polynomial for the quiver Grassmannian

$$\mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^m\big).$$

In Chapter 4, we have seen that the dimension of a cell equals the number of holes (i.e. white dots) below the starting points of the segments in the successor closed subquiver corresponding to the cell. Now we describe how to compute this number directly from the length of the segments.

Proposition 5.8. The function

$$\begin{aligned} \mathrm{h} : \mathcal{C}_{xN}^{(m)}\big(\Delta_1, \mathrm{I}_N\big) &\to \mathbb{Z} \\ \mathbf{p} &\longmapsto \mathrm{h}(\mathbf{p}) := \sum_{j\in[m]} \mathrm{h}_j(\mathbf{p}) \end{aligned}$$

where

$$\begin{aligned} \mathrm{h}_j(\mathbf{p}) := \max\big\{0, (m-1)p_j - j + 1\big\} \\ - \Big(\sum_{i>j} \min\{p_i, p_j\} + \sum_{i<j} \min\big\{p_i, \max\{p_j - 1, 0\}\big\} \Big) \end{aligned}$$

computes the dimension of the corresponding cells in the quiver Grassmannian

$$\mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^m\big).$$

Proof. It remains to show that the function $\mathrm{h}_j(\mathbf{p})$ counts the number of holes below the starting point of the j-th segment in the coefficient quiver. The length of this segment is given by the number p_j. Accordingly

$$\max\big\{0, m - j + (m-1)(p_j - 1)\big\} = \max\big\{0, (m-1)p_j - j + 1\big\}$$

is the number of points below the starting point of the j-th segment which do not belong to the segment itself. From this we have to subtract all points which belong to other segments. For this we have to distinguish segments with endpoints above the endpoint of the j-th segment and segments with endpoints below the endpoint of the j-th segment. This corresponds to the cases $i < j$ and $i > j$. For $i > j$, the number of points from the i-th segments which are below the starting point of the j-th segment is given by

$$\min\{p_i, p_j\}.$$

For $i < j$, the starting point of the i-th segment is above the starting point of the j-th segments if both segments have the same length. Hence the number of points from the i-th segment which are below the starting point of the j-th segment is given by

$$\min\big\{p_i, \max\{p_j - 1, 0\}\big\}.$$

□

Combining this dimension function h with the parametrisation of the set of cells $\mathcal{C}_{xN}^{(x+y)}$ we obtain the subsequent description of the Poincaré polynomials.

PROPOSITION 5.9. The Poincaré polynomial $\mathrm{p}_{x,y,N}(t)$ for the quiver Grassmannian

$$\mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^{x+y}\big)$$

is given by

$$\mathrm{p}_{x,y,N}(t) = \sum_{\mathbf{p}\,\in\,\mathcal{C}_{xN}^{(x+y)}} t^{\mathrm{h}(\mathbf{p})}.$$

REMARK. The parametrisation of the cells and the function to compute their dimension we have introduced in this section is suitable to be implemented as a computer program as done in Appendix B.2. With a program following this approach we computed all the examples for Poincaré polynomials in this chapter and more examples are given in Appendix C.2.

5.2. The Affine Grassmannian

DEFINITION 5.10. Let $\widehat{G}$ be the Kac-Moody group corresponding to the affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$. For the maximal parahoric subgroup $P = P_0$ of $\widehat{G}$ the **affine Grassmannian** is defined as

$$\mathrm{Gr}\big(\,\widehat{\mathfrak{g}}\,\big) := \widehat{G}/P_0.$$

In this chapter we study the affine Grassmannian for the affine Kac-Moody Lie algebra $\widehat{\mathfrak{gl}}_n$ and refer to it as the affine Grassmannian. In order to identify its approximations with quiver Grassmannians we need a different description of the affine Grassmannian which is closer to the subspace interpretation of the classical Grassmannian. It is possbile to identify the affine Grassmannian with the set of lattices as shown for example in the survey by U. Görtz [**37**]. But there is an other approach which realises the affine Grassmannian by a construction which is even closer to the classical interpretation. This is based on the embedding $\widehat{\mathfrak{gl}}_n \subset \mathfrak{gl}_\infty$ and is described in the article by V. Kac and D. Peterson [**45**].

Let V be an infinite dimensional vector space over $\mathbb{C}$ with basis vectors v_i for $i \in \mathbb{Z}$ and consider the subspaces

$$V_\ell := \mathrm{span}\big(v_\ell, v_{\ell-1}, v_{\ell-2}, \dots\big)$$

which are infinite in the direction of the negative indices.

The **Sato Grassmannian** SGr_m is defined as

$$\mathrm{SGr}_m := \Big\{U \subset V \ : \text{ There exists a } \ell < m \text{ s.t. } V_\ell \subset U \text{ and } \dim U/V_\ell = m - \ell\,\Big\}.$$

The points in this Grassmannian are vector subspaces of V which are infinite in the direction of negative indices. But for each space there exists a number ℓ such that the part of the vector space living above the vector v_ℓ is finite dimensional. More precisely every point in SGr_m can be described as

$$U = \mathrm{span}\Big(\big\{v_i : i \leq \ell\big\} \cup \big\{w_k : k \in I\big\}\Big)$$

where $|I| = m - \ell$ and the w_k are linear independent combinations of the v_i's with $i > \ell$. For example

$$V_m \text{ and } \mathrm{span}\Big(\big\{v_i : i \leq m-3\big\} \cup \big\{v_{m+1}, v_{m+5}, v_{m+13}\big\}\Big)$$

describe points in the Sato Grassmannian SGr_m.

There exists an alternative parametrisation of the affine Grassmannian as a subset in the Sato Grassmannian SGr_0. Let

$$s_n : V \to V;\ v_i \mapsto v_{i+n}$$

be a shift of indices by n.

PROPOSITION 5.11. As a subset of the Sato Grassmannian SGr_0 the affine Grassmannian is described as

$$\mathrm{Gr}\big(\,\widehat{\mathfrak{gl}}_n\,\big) \cong \big\{\, U \in \mathrm{SGr}_0 :\ U \subset s_n U \,\big\}.$$

PROOF. In [**30**] E. Feigin, M. Finkelberg and M. Reineke introduce the affine Grassmannian as

$$\mathrm{Gr}\big(\,\widehat{\mathfrak{gl}}_n\,\big) \cong \big\{\, U \in \mathrm{SGr}_0 :\ U \subset t^{-1} U \,\big\}.$$

This description goes back to V. Kac and D. Peterson who studied representations of $\widehat{\mathfrak{gl}}_n$ and $\mathfrak{gl}_\infty$ in [**45**]. The infinite dimensional vectorspace V is identified with $\mathbb{C}^n \otimes \mathbb{C}[t, t^{-1}]$ via

$$v_{n(k-1)+j} = e_j \otimes t^{-k}$$

where $\{e_i : i \in [n]\}$ is the standard basis of $\mathbb{C}^n$. With this identification of vectorspaces the multiplication by t^{-1} in $\mathbb{C}^n \otimes \mathbb{C}[t, t^{-1}]$ corresponds to an index shift by n in the basis of V. □

5.3. Finite Approximations by Quiver Grassmannians for the Loop

In this section we define the Feigin-degenerate affine Grassmannians $\mathrm{Gr}^a\big(\,\widehat{\mathfrak{gl}}_n\,\big)$ and approximate them by quiver Grassmannians for the loop quiver. This construction was introduced by E. Feigin, M. Finkelberg and M. Reineke in [**30**].

On V we define the projection

$$\mathrm{pr_k} : V \to V;\ v_i \mapsto \begin{cases} 0 & \text{if } i = k \\ v_i & \text{otherwise.} \end{cases}$$

DEFINITION 5.12. The **degenerate affine Grassmannian** is defined as

$$\mathrm{Gr}^a\big(\,\widehat{\mathfrak{gl}}_n\,\big) := \big\{\, U \in \mathrm{SGr}_0 :\ \mathrm{pr}\, U \subset s_n U \,\big\}$$

where

$$\mathrm{pr} := \mathrm{pr}_1 \circ \mathrm{pr}_2 \circ \cdots \circ \mathrm{pr}_n.$$

We refer to this degeneration as the Feigin degeneration of the affine Grassmannian if we have to distinguish between different degenerations because it is defined analogous to the interpretation of the Feigin degeneration of the classical flag variety by E. Feigin [**29**, Theorem 0.1].

For a parameter $N \in \mathbb{N}$ the **finite approximation** of the degenerate affine Grassmannian is defined as

$$\mathrm{Gr}_N^a\big(\,\widehat{\mathfrak{gl}}_n\,\big) := \Big\{\ U \in \mathrm{Gr}^a\big(\,\widehat{\mathfrak{gl}}_n\,\big)\ :\ V_{-nN} \subset U \subset V_{nN}\ \Big\}.$$

REMARK. In [**30**, Definition 2.2] the approximations are defined by the condition

$$\mathbb{C}^n \otimes t^N\mathbb{C}[t] \subset U \subset \mathbb{C}^n \otimes t^{-N}\mathbb{C}[t].$$

This is equivalent to our description because of the identification of basis we made above. Namely the basis vector v_{nN} corresponds to $e_n \otimes t^{-N}$ since

$$n(N-1)+n = nN.$$

This is the biggest possible index in the basis of V which the basis elements of $\mathbb{C}^n \otimes t^{-N}\mathbb{C}[t]$ can have. The basis vector v_{-nN} corresponds to $e_n \otimes t^N$ because

$$n(-N-1)+n = -nN.$$

This is the biggest possible index for the basis elements of $\mathbb{C}^n \otimes t^N\mathbb{C}[t]$.

Finite approximations of Sato Grassmannians are the same as classical Grassmannians.

PROPOSITION 5.13. For $\ell \in \mathbb{N}$ and $m \leq \ell$ the approximation

$$\mathrm{SGr}_{m,\ell} := \Big\{ U \in \mathrm{SGr}_m\ :\ V_{-\ell} \subset U \subset V_\ell \Big\}$$

of the Sato Grassmannian SGr_m is isomorphic to the classical Grassmannian

$$\mathrm{Gr}_{m+\ell}(2\ell) = \Big\{ U \subset \mathbb{C}^{2\ell} :\ \dim U = m+\ell \Big\}.$$

PROOF. Let U be a point in the classical Grassmannian $\mathrm{Gr}_{m+\ell}(2\ell)$. It is of the form

$$U = \mathrm{span}\big(w_1, w_2, \dots, w_{m+\ell}\big)$$

where the w_j's are linear independent linear combinations of the standard basis vectors e_i of $\mathbb{C}^{2\ell}$. Namely every w_j is of the form

$$w_j = \sum_{i=1}^{2\ell} \lambda_i\, e_i$$

with $\lambda_i \in \mathbb{C}$. The vectors

$$\tilde{w}_j := \sum_{i=1}^{2\ell} \lambda_i\, v_{\ell+1-i} = \sum_{i=-\ell+1}^{\ell} \lambda_{\ell+1-i}\, v_i$$

are linearly independent in V. We define

$$\tilde{U} := \mathrm{span}\Big(\big\{v_i : i \leq -\ell\big\} \cup \big\{\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_{m+\ell}\big\}\Big)$$

which describes a point in the approximation $\mathrm{SGr}_{m,\ell}$. For a different choice of the basis of U we obtain the same point $\tilde{U}$ in the approximation of the Sato Grassmannian. Moreover this map

$$\phi : \mathrm{Gr}_{m+\ell}(2\ell) \to \mathrm{SGr}_{m,\ell};\ U \mapsto \tilde{U}$$

is injective.

Let $\tilde{U}$ be a point in $\mathrm{SGr}_{m,\ell}$. By definition we have $V_{-\ell} \subset U \subseteq V_\ell$ and $\dim U/V_{-\ell} = m + \ell$. This means that we can write it as

$$\tilde{U} = \mathrm{span}\Big(\big\{v_i : i \leq -\ell\big\} \cup \big\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m+\ell}\big\}\Big)$$

where the $\tilde{u}_j$'s are linear independent linear combinations of the basis vectors v_i of V with $-\ell < i \leq \ell$. Hence every $\tilde{u}_j$ is of the form

$$\tilde{u}_j = \sum_{i=-\ell+1}^{\ell} \mu_i v_i.$$

Accordingly the vectors

$$u_j := \sum_{i=-\ell+1}^{\ell} \mu_i e_{\ell+1-i} = \sum_{i=1}^{2\ell} \mu_{\ell+1-i} e_i$$

are linearly independent in $\mathbb{C}^{2\ell}$ and

$$U := \mathrm{span}\big(u_1, u_2, \ldots, u_{m+\ell}\big)$$

describes a point in the classical Grassmannian $\mathrm{Gr}_{m+\ell}(2\ell)$. This is again independent of the choice of the basis for $\tilde{U}$. The map

$$\psi : \mathrm{SGr}_{m,\ell} \to \mathrm{Gr}_{m+\ell}(2\ell);\ \tilde{U} \to U$$

is injective and inverse to the map ϕ. □

This observation about the shape of approximations of Sato Grassmannians allows to identify approximations of the affine Grassmannian with quiver Grassmannians for the loop quiver.

PROPOSITION 5.14. For every $N \in \mathbb{N}$ the approximation $\mathrm{Gr}^a_N(\widehat{\mathfrak{gl}}_n)$ of the degenerate affine Grassmannian is isomorphic to the quiver Grassmannian

$$\mathrm{Gr}^{A_N}_{Nn}\big(A_N \otimes \mathbb{C}^{2n}\big)$$

where $A_N = \mathbb{C}[t]/(t^N)$ is the truncated path algebra for the loop quiver with paths of length at most N.

For the loop quiver there is exactly one indecomposable bounded injective and one indecomposable bounded projective representation. Both are isomorphic to the truncated path algebra A_N of the loop quiver.

PROOF. The vector space R corresponding to the quiver representation $A_N \otimes \mathbb{C}^{2n}$ is $2nN$-dimensional. We label its standard basis vectors by r_i for $i \in [2nN]$. The map $M_\alpha : R \to R$ which corresponds to this quiver representation is given by

$$M_\alpha : r_i \mapsto \begin{cases} 0 & \text{if } i + n > 2nN \\ 0 & \text{if } i \in \{nN, nN-1, \ldots, nN-n+1\} \\ r_{i+n} & \text{otherwise} \end{cases}$$

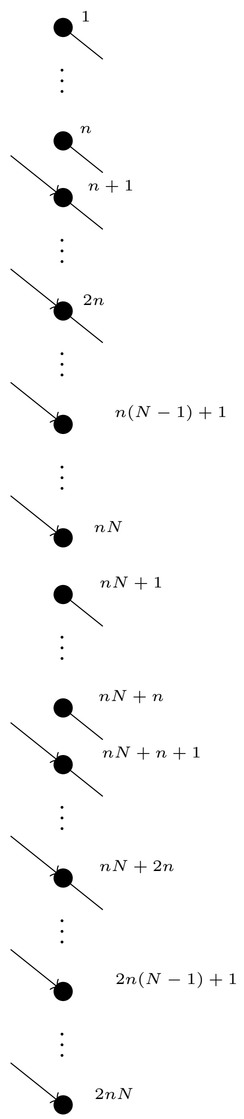
In other words

$$M_\alpha = s_n \circ \mathrm{pr}_{nN} \circ \mathrm{pr}_{nN-1} \circ \cdots \circ \mathrm{pr}_{nN-n+1}.$$

This description of the map M_α is obtained as follows. A_N is isomorphic to the bounded projective representation $P_1^{(N)}$ and bounded injective representation $I_1^{(N)}$ of the loop quiver. Hence we can view $A_N \otimes \mathbb{C}^{2n}$ as

$$I_1^{(N)} \otimes \mathbb{C}^n \oplus P_1^{(N)} \otimes \mathbb{C}^n$$

which is analogous to the general case studied in the section about the quiver Grassmannians for the equioriented cycle. In the coefficient quiver of this representation we arrange the segments corresponding to the injective summands above the segments of the projective summands, i.e.



In this picture the arrows go from i to $i + n$ if both indices are smaller than nN or both are strictly bigger than nN. From this picture we obtain the map M_α as introduced above.

The points in the quiver Grassmannian can be identified with points U in the classical Grassmannian $\mathrm{Gr}_{nN}(R)$ such that $M_\alpha(U) \subset U$. By the identification of points in Sato Grassmannians and classical Grassmannians as above, this corresponds to the points $\tilde{U}$ in the approximation $\mathrm{SGr}_{0,nN}$ such that

$$s_{-n} \circ \mathrm{pr}\, \tilde{U} \subset \tilde{U}.$$

This condition is equivalent to the description of the approximation of the affine Grassmannian as subset of the Sato Grassmannian SGr_0. □

5.4. Linear Degenerations

In this section we describe linear degenerations of the affine Grassmannian following the degeneration approach by G. Cerulli Irelli, X. Fang, E. Feigin, G. Fourier and M. Reineke as introduced in [**19**]. For a map $f : V \to V$ define the f-**degenerate affine Grassmannian** as

$$\mathrm{Gr}^f\big(\,\widehat{\mathfrak{gl}}_n\,\big) := \big\{U \in \mathrm{SGr}_0 : f(U) \subset s_n\, U\big\}.$$

If f is a linear map, the degeneration is called linear. The degenerations we study here are choosen such that the quiver Grassmannians approximating them can be studied using the methods developed in this thesis. Even for this small class of maps the degenerations behave very different from the linear degenerations of the classical flag variety studied in [**19**]. For this reason we do not consider a more general class of degenerations here.

For

$$f = \mathrm{pr}_1 \circ \mathrm{pr}_2 \circ \cdots \circ \mathrm{pr}_n =: \mathrm{pr}$$

we obtain the Feigin degeneration of the affine Grassmannian

$$\mathrm{Gr}^a\big(\,\widehat{\mathfrak{gl}}_n\,\big) := \big\{U \in \mathrm{SGr}_0 : \mathrm{pr}\; U \subset s_n\, U\big\}$$

as studied in [**30**].

The goal of this section is the characterisation of the intermediate degenerations between the Feigin degeneration and the affine Grassmannian. For integers n and k the set of k-element subsets of $[n] := \{1, 2, \dots, n\}$ is defined as

$$\binom{[n]}{k} := \Big\{I \subset [n] : |I| = k \text{ and } i_p \neq i_q \text{ for all } p, q \in [k] \text{ with } p \neq q\Big\}.$$

Take an index set $I \in \binom{[n]}{k}$ with $I = \{i_1, \dots, i_k\}$ and define the function

$$\mathrm{pr}_I := \mathrm{pr}_{i_1} \circ \mathrm{pr}_{i_2} \circ \cdots \circ \mathrm{pr}_{i_k}.$$

Let $I = [k]$ for $k \in [n]$ and define

$$\mathrm{Gr}^k\big(\,\widehat{\mathfrak{gl}}_n\,\big) := \mathrm{Gr}^{\mathrm{pr}_I}\big(\,\widehat{\mathfrak{gl}}_n\,\big)$$

which is called the standard partial degeneration to the parameter k.

LEMMA 5.15. *For every linear map $f : V \to V$ with* $\operatorname{corank} f = k$ *there is an isomorphism of degenerate affine Grassmannians*

$$\mathrm{Gr}^f\big(\,\widehat{\mathfrak{gl}}_n\,\big) \cong \mathrm{Gr}^k\big(\,\widehat{\mathfrak{gl}}_n\,\big).$$

The proof of this Lemma will be divided into several parts. At first we want to construct approximations of the standard partial degeneration to the parameter k. Later we prove that all partial degenerate affine Grassmannians admit approximations by quiver Grassmannians for the loop quiver and identify their approximations with the approximations of the standard partial degeneration.

The approximations by quiver Grassmannians for the loop quiver exist for a much bigger class of degenerations than introduced above. But the examination of the degenerations we introduce here shows that already for this class we lose some properties of the corresponding quiver Grassmannians which were used in the previous chapters. Namely we can not describe the approximations by quiver Grassmannians containing subrepresentations of representations which only consist of injective representations of the loop. Hence we can not apply Theorem 2.3 and can not study the variety of quiver representations to understand the geometric properties of the quiver Grassmannian.

For this reason we restrict our further study to the linear degenerations of the affine Grassmannian between the Feigin degeneration and the non-degenerate affine Grassmannian. Some of the results about the geometric properties of the corresponding quiver Grassmannians as introduced in the first section of this chapter are generalised to this setting later.

LEMMA 5.16. *For every $N \in \mathbb{N}$ the approximation $\mathrm{Gr}^k_N(\widehat{\mathfrak{gl}}_n)$ of the degenerate affine Grassmannian is isomorphic to the quiver Grassmannian*

$$\mathrm{Gr}^{A_{2N}}_{Nn}\big(A_N \otimes \mathbb{C}^{2k} \oplus A_{2N} \otimes \mathbb{C}^{n-k}\big).$$

PROOF. The vectorspace R corresponding to the quiver representation

$$M^k_N := A_N \otimes \mathbb{C}^{2k} \oplus A_{2N} \otimes \mathbb{C}^{n-k}$$

is $2nN$-dimensional. It is possible to arrange the segments in the coefficient quiver of M^k_N such that the map M_α corresponding to this representation is given by

$$M_\alpha = s_n \circ \mathrm{pr}_{nN} \circ \mathrm{pr}_{nN-1} \circ \cdots \circ \mathrm{pr}_{nN-k+1}.$$

The nN-dimensional subrepresentations of M^k_N are described by the points U in the Grassmannian $\mathrm{Gr}_{nN}(2nN)$ satisfying $M_\alpha(U) \subset U$. This Grassmannian is isomorphic to the approximation $\mathrm{SGr}_{0,N}$ of the Sato Grassmannian SGr_0. Following the isomorphism of Grassmannians the map M_α corresponds to the map

$$\tilde{M}_\alpha = s_{-n} \circ \mathrm{pr}_1 \circ \mathrm{pr}_2 \circ \cdots \circ \mathrm{pr}_k : V \to V$$

which parametrises the points in the approximation of the affine Grassmannian by the condition $\tilde{M}_\alpha(U) \subset U$ for $U \in \mathrm{SGr}_{0,N}$. □

With the same methods we can construct the approximations of the other degenerations where the linear map is projection.

PROPOSITION 5.17. Let $I, J \in \binom{[n]}{k}$, then

$$\mathrm{Gr}^{\mathrm{pr}_I}\big(\widehat{\mathfrak{gl}}_n\big) \cong \mathrm{Gr}^{\mathrm{pr}_J}\big(\widehat{\mathfrak{gl}}_n\big).$$

PROOF. Without loss of generality we can assume that $I = [k]$. For $N \geq 1$ let $V^{(N)}$ be the subspace of V which is spanned by the basis vectors

$$\big\{v_i : -nN < i \leq nN\big\}.$$

The maps

$$s_{-n} \circ \mathrm{pr}_1 \circ \mathrm{pr}_2 \circ \cdots \circ \mathrm{pr}_k : V^{(N)} \to V^{(N)}$$

and

$$s_{-n} \circ \mathrm{pr}_{i_1} \circ \mathrm{pr}_{i_2} \circ \cdots \circ \mathrm{pr}_{i_k} : V^{(N)} \to V^{(N)}$$

can be realised by different arrangements of the segments of the quiver representation

$$M_N^k := A_N \otimes \mathbb{C}^{2k} \oplus A_{2N} \otimes \mathbb{C}^{n-k}$$

hence the quiver Grassmannians providing the approximations of the partial degenerate affine Grassmannians are isomorphic and the isomorphism for $N = 1$ extends to the isomorphism for all bigger approximations. □

In the same way we prove the isomorphism for the other partial degenerations of the affine Grassmannian.

Proof of Lemma 5.15. Let $N \in \mathbb{N}$ be the smallest number such that the corank of f as endomorphism of V is the same as the corank of f as an endomorphism of $V^{(N)}$. Then there exists a matrix $g \in \mathrm{GL}_{2nN}(\mathbb{C})$ such that

$$gfg^{-1} = \mathrm{pr}_1 \circ \mathrm{pr}_2 \circ \cdots \circ \mathrm{pr}_k.$$

Hence the quiver representations whose quiver Grassmannians provide the approximations of

$$\mathrm{Gr}_N^k\big(\widehat{\mathfrak{gl}}_n\big) \quad \text{and} \quad \mathrm{Gr}_N^f\big(\widehat{\mathfrak{gl}}_n\big)$$

are isomorphic. This isomorphism extends to an isomorphism for all bigger approximations. □

From now on we will restrict our study to the degenerations $\mathrm{Gr}^k\big(\widehat{\mathfrak{gl}}_n\big)$ because their properties translate to all other degenerations in the corresponding isomorphism class of partial degenerate affine Grassmannians.

5.5. Ind-Variety Structure

The affine Grassmannian and its linear approximations are in contrast to the classical case not finite dimensional. Hence we have to give some structure to their finite dimensional approximation in order to lift geometric properties from the finite approximations to the infinite dimensional object.

Definition 5.18. A set X is called **ind-variety** if there is a filtration of finite dimensional varieties $X_0 \subseteq X_1 \subseteq X_2 \dots$ such that

(1) $\bigcup_{i \geq 0} X_i = X$,
(2) $X_i \hookrightarrow X_{i+1}$ is a closed embedding for all $i \in \mathbb{Z}_{\geq 0}$.

In this section we construct closed embeddings

$$\Phi_N^k : \mathrm{Gr}_{Nn}^{A_{2N}}\big(A_N \otimes \mathbb{C}^{2k} \oplus A_{2N} \otimes \mathbb{C}^{n-k}\big) \to \mathrm{Gr}_{(N+1)n}^{A_{2N+2}}\big(A_{N+1} \otimes \mathbb{C}^{2k} \oplus A_{2N+2} \otimes \mathbb{C}^{n-k}\big).$$

These maps provide the ind-variety structure for the finite approximations of the partial degenerate affine Grassmannian by quiver Grassmannians for the loop quiver. For the affine Grassmannian and the Feigin degeneration these maps preserve the dimension of the cells in the quiver Grassmannians. This implies that the ind-topology and the Zariski topology on the ind-variety coincide [**71**, Proposition 7].

5.5.1. Ind-Variety Structure of the Feigin Degeneration of the Affine Grassmannian. Before we introduce the construction of this map in the general setting, we consider it for the special case of the Feigin degeneration, i.e.

$$\Phi_N^a : \mathrm{Gr}_{Nn}^{A_N}\big(A_N \otimes \mathbb{C}^{2n}\big) \to \mathrm{Gr}_{(N+1)n}^{A_{N+1}}\big(A_{N+1} \otimes \mathbb{C}^{2n}\big)$$

For the definition of this map we need an explicit description for the coordinates of the points in the quiver Grassmannians. The vector space V corresponding to the quiver representation $M_N := A_N \otimes \mathbb{C}^{2n}$ is $2nN$-dimensional and we denote its basis by

$$\mathcal{B} := \{v_1, v_2, \dots, v_{2nN}\}$$

following the notation in Section 5.1.4.

Each point p in the quiver Grassmannian corresponds to a nN-dimensional subspace of V which is compatible with the map M_α corresponding to the quiver representation M_N. It can be written as

$$p = \mathrm{Span}\big\{w_1, \dots, w_{nN}\big\}$$

with $w_k \in \mathbb{C}^{2nN}$. Below we work out an explicit description of the vectors w_k following the construction in the proof of Theorem 4.10.

Proposition 5.19. The cells of the quiver Grassmannian

$$\mathrm{Gr}_{Nn}^{A_N}\big(A_N \otimes \mathbb{C}^{2n}\big)$$

are in bijection with the set

$$\mathcal{I}_{nN}(2nN) := \big\{I \subset [2nN] : |I| = nN \text{ and } k + 2n \in I \text{ if } k \in I\big\}.$$

Proof. The cells are in bijection with successor closed subquivers on nN points in the coefficient quiver of $A_N \otimes \mathbb{C}^{2n}$. By definition, every point in the coefficient quiver is labelled by some basis vector v_k. Hence, any full subquiver in the coefficient quiver of M_N can be described by a subset $I \subseteq [2nN]$. The arrows in the coefficient quiver of M_N are going form the point labelled by v_k to the point labelled by v_{k+2n}. Accordingly a subquiver is successor closed if and only if $k \in I$ implies that $k + 2n \in I$ if the second number is not bigger than $2nN$. The correct dimension of the subrepresentations in the quiver Grassmannian above is obtained with the condition $|I| = nN$. □

To an index set $I \in \mathcal{I}_{nN}(2nN)$ we assign the torus fixed point

$$p_I := \mathrm{Span}\{v_k : k \in I\}.$$

Following the computation in the proof of Theorem 4.10 we obtain that the points in the attracting set of p_I can be described as $\mathrm{Span}\{w_1, \dots, w_{nN}\}$ where

$$w_s = v_{k_s} + \sum_{j > k_s, j \notin I} \lambda_{j,s} v_j$$

with $\lambda_{j+2n,t} = \lambda_{j,s} \in \mathbb{C}$ whenever M_α maps v_{k_s} to v_{k_t}.

The dimension of the cell $c(p_I)$ is obtained as the number of independent parameters in the set

$$\Big\{\lambda_{j,s} : k_s \in I \text{ and } j > k_s, j \notin I\Big\}.$$

This number equals the number of holes below the starting points of the segments corresponding to I in the coefficient quiver of M_N.

The coefficients describing a point in the quiver Grassmannian are collected in a matrix

$$M(\lambda) \in M_{2nN,nN}(\mathbb{C})$$

where the s-th column of the matrix has the entries $\lambda_{j,s}$ from the description of the points as above. We set $\lambda_{j,s} = 0$ if the vector v_j does not turn up in the above sum. Then entry $\lambda_{k_s,s}$ is set to one because v_{k_s} turns up with coefficient one in the above summation.

We define the map

$$\Psi_N : M_{2nN,nN}(\mathbb{C}) \to M_{2n(N+1),n(N+1)}(\mathbb{C})$$

where the matrix $\tilde{M} := \Psi_N(M)$ is defined by

$$\tilde{m}_{p,q} := \begin{cases} m_{p-n,q} & \text{if } n < p \leq 2nN + n \text{ and } q \in [nN] \\ 1 & \text{if } q > nN \text{ and } p - 2nN = q - nN \\ 0 & \text{otherwise.} \end{cases}$$

This matrix has a block structure of the following shape

$$\tilde{M} = \begin{pmatrix} \mathbf{0}_{n,nN} & \mathbf{0}_{n,n} \\ M & \mathbf{0}_{nN,n} \\ \mathbf{0}_{n,nN} & \mathrm{id}_n \end{pmatrix}$$

where $\mathbf{0}_{p,q}$ is a $p \times q$ matrix with all entries equal to zero and id_n is the $n \times n$ identity matrix.

Using this map we define the closed embeddings for the ind-variety structure on the affine Grassmannian. Therefore it needs to send points of the smaller quiver Grassmannian to points in the bigger quiver Grassmannian.

PROPOSITION 5.20. Let $M := M(\lambda) \in M_{2nN,nN}(\mathbb{C})$ describe a point in the quiver Grassmannian

$$\mathrm{Gr}_{nN}^{A_N}\big(A_N \otimes \mathbb{C}^{2n}\big).$$

Then $\Psi_N(M)$ describes a point in the quiver Grassmannian

$$\mathrm{Gr}_{n(N+1)}^{A_{N+1}}\big(A_{N+1} \otimes \mathbb{C}^{2n}\big).$$

PROOF. Each point in the quiver Grassmannian

$$\mathrm{Gr}_{Nn}^{A_N}\big(A_N \otimes \mathbb{C}^{2n}\big).$$

is contained in some cell which is corresponding to an index set $I \in \mathcal{I}_{nN}(2nN)$. Hence the point can be described as $\mathrm{Span}\{w_1, \ldots, w_{nN}\}$ where

$$w_s = v_{k_s} + \sum_{j > k_s, j \notin I} \lambda_{j,s} v_j$$

with $\lambda_{j+2n,t} = \lambda_{j,s} \in \mathbb{C}$ whenever M_α maps v_{k_s} to v_{k_t}.

Let

$$\tilde{\mathcal{B}} := \big\{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{2n(N+1)}\big\}$$

be a basis of the vector space $\tilde{V}$ which corresponds to the quiver representation M_{N+1}. The image of the vector w_s under the map Ψ_N is given by

$$\tilde{w}_s := \Psi_N(w_s) = \tilde{v}_{k_s+n} + \sum_{j > k_s, j \notin I} \lambda_{j,s} \tilde{v}_{j+n}$$

with $\lambda_{j+2n,t} = \lambda_{j,s}$ if M_α sends v_{k_s} to v_{k_t}. For $k \in [n(N+1)] \setminus [nN]$, the map Ψ_N generates the additional vectors $\tilde{w}_k = \tilde{v}_{n(N+1)+k}$.

Accordingly the image of $\mathrm{Span}\{w_1, \dots, w_{nN}\}$ under Ψ_N is given as

$$\mathrm{Span}\big\{\tilde{w}_1, \dots, \tilde{w}_{nN}, \tilde{v}_{2nN+n+1}, \dots, \tilde{v}_{2n(N+1)}\big\}.$$

We define the index set $\tilde{I} \subset [2n(N+1)]$ which contains the indices of the first non-zero coefficient in the rows of the matrix $\Psi_N(M(\lambda))$. By the shape of the map Ψ_N we know that the set $\tilde{I}$ is obtained from the index set I as

$$\tilde{I} = \{k + n : k \in I\} \cup \{2nN + n + 1, 2nN + n + 2, \dots, 2n(N+1)\}.$$

Now we check that

$$\tilde{I} \in \mathcal{I}_{n(N+1)}\big(2n(N+1)\big),$$

i.e. the index set $\tilde{I}$ describes a cell in the quiver Grassmannian

$$\mathrm{Gr}^{A_{N+1}}_{n(N+1)}\big(A_{N+1} \otimes \mathbb{C}^{2n}\big).$$

By definition we know that $\tilde{I}$ has the right cardinality. The index set I satisfies that $k + 2n \in I$ if $k \in I$. Thus for every $k \in I$ there exists an $\ell \in \mathbb{N}$ such that $k+2n\ell \in I$ and $k+2n\ell \in [2nN]\setminus[2n(N-1)]$. This means that $k+2n(\ell+1) > 2nN$ such that we do not have to consider this repetition for the index set $I \subset [2nN]$.

In the index set $\tilde{I}$ this corresponds to $k + n + 2n(\ell + 1) > 2nN + n$. By construction this element is contained in the index set $\tilde{I}$ because it contains all indices bigger than $2nN + n$. Moreover it is the largest element which has to be contained in $\tilde{I}$ since $k + n + 2n(\ell + 2) > 2n(N + 1)$.

For smaller ℓ the index $k+2n\ell$ is included in $\tilde{I}$ if k is included in $\tilde{I}$ because this part of the index set $\tilde{I}$ is obtained as a shift of I by n and I satisfies this condition. Accordingly the index set $\tilde{I}$ satisfies the requirements to be included in

$$\mathcal{I}_{n(N+1)}\big(2n(N+1)\big).$$

It remains to show that the point $\mathrm{Span}\{\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_{n(N+1)}\}$ is included in the attracting set of the fixed point

$$p_{\tilde{I}} := \mathrm{Span}\Big\{\tilde{v}_k : k \in \tilde{I}\Big\}.$$

The points q in the cell of $p_{\tilde{I}}$ are of the form $q = \mathrm{Span}\{\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{n(N+1)}\}$ where

$$\tilde{u}_s = \tilde{v}_{\tilde{k}_s} + \sum_{\tilde{j} > \tilde{k}_s, \tilde{j} \notin \tilde{I}} \tilde{\lambda}_{\tilde{j},s} \tilde{v}_{\tilde{j}}.$$

For $\tilde{k}_s > 2nN + n$ there are no indices $\tilde{j} > \tilde{k}_s$ with $\tilde{j} \notin \tilde{I}$ because the index set $\tilde{I}$ contains all indices bigger than $2nN + n$. Thus we have $\tilde{u}_s = \tilde{v}_{\tilde{k}_s}$ for these indices and there are n-many of them. This matches the generators

$$\tilde{w}_{nN+1}, \tilde{w}_{nN+2}, \dots, \tilde{w}_{n(N+1)}$$

of the image of the point from the smaller quiver Grassmannian.

The other $\tilde{k}_s \in \tilde{I}$ are obtained as $k_s + n$ for $k_s \in I$. Hence the condition

$$\tilde{j} > \tilde{k}_s, \tilde{j} \notin \tilde{I}$$

is equivalent to

$$j > k_s, j \notin I$$

for $j = \tilde{j} - n$. With this index shift we can rewrite the vectors as

$$\tilde{u}_s = \tilde{v}_{k_s+n} + \sum_{j > k_s, j \notin I} \tilde{\lambda}_{j+n,s} \tilde{v}_{j+n}.$$

By setting $\tilde{\lambda}_{j+n,s} = \lambda_{j,s}$ we obtain that the remaining $\tilde{w}_s$ are of this form. Therefore the point

$$\mathrm{Span}\big\{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{n(N+1)}\big\}$$

is contained in the attracting set of the torus fixed point $p_{\tilde{I}}$. □

It remains to show that this map preserves the dimension of the cells in the approximations.

PROPOSITION 5.21. The cell in the approximation $\mathrm{Gr}^a_N(\widehat{\mathfrak{gl}}_n)$ of the Feigin-degenerate affine Grassmannian which is parametrised by the index set I is of the same dimension as the cell in $\mathrm{Gr}^a_{N+1}(\widehat{\mathfrak{gl}}_n)$ which corresponds to the index set $\tilde{I}$.

PROOF. In the proposition above we have seen that both index sets correspond to cells in certain quiver Grassmannians for the loop quiver. Because of the $\mathbb{C}^*$-action the cells of these quiver Grassmannians are in bijection with successor closed subquivers in the coefficient quiver of the representation $A_N \otimes \mathbb{C}^{2n}$ respective $A_{N+1} \otimes \mathbb{C}^{2n}$. The coefficient quiver of $A_N \otimes \mathbb{C}^{2n}$ has $2nN$ points. We can assign subquiver to an index set $I \in \mathcal{I}_{nN}(2nN)$ by colouring the points corresponding to the indices $k \in I$ black and the remaining ones white. The properties of the index sets in $\mathcal{I}_{nN}(2nN)$ guarantee that this subquiver is successor closed. This correspondence is bijective as described earlier in this section.

In the coefficient quiver parametrisation we can compute the dimension of cells by counting the holes below the starting points of the subsegments. The way how we obtain $\tilde{I}$ from the index set I guaranties that the number of holes below each segment stays the same by passing from I to $\tilde{I}$. More precisely we shift all the segments corresponding to I by n and add only black dots below the lowest hole of the subquiver corresponding to I. □

In the subsequent example we study the shape of the map between the approximations in the parametrisation using coefficient quivers. There exists an equivalent characterisation of the index sets $I \in \mathcal{I}_{nN}(2nN)$ which helps to compute them.

PROPOSITION 5.22. The cells of the quiver Grassmannian

$$\mathrm{Gr}^{A_N}_{nN}\big(A_N \otimes \mathbb{C}^{2n}\big)$$

are parametrised by the set

$$\mathcal{I}_{N,nN}(2n) := \Big\{ I = (I_\ell)_{\ell \in [N]} : I_1 \subseteq I_2 \subseteq \cdots \subseteq I_N \subseteq [2n] \text{ and } \sum_{\ell=1}^{N} |I_\ell| = nN \Big\}.$$

PROOF. The cells of this quiver Grassmannian are parametrised by the set $\mathcal{I}_{nN}(2nN)$. To a tuple $I = (I_\ell)_{\ell=1}^N$ we assign the index set

$$J := \bigcup_{\ell \in [N]} \big\{ k + 2n(\ell-1) : k \in I_\ell \big\}$$

which is contained in $\mathcal{I}_{nN}(2nN)$ because the shifts of the indices do not change the amount of all indices and the index sets satisfy $I_\ell \subseteq I_{\ell+1}$ for all $\ell \in [N-1]$.

Starting with an index set $J \in \mathcal{I}_{nN}(2nN)$ we define

$$I_\ell := \Big\{ k - 2n(\ell-1) : k \in J \text{ and } k - 2n(\ell-1) \in [2n] \Big\}.$$

The elements of the set J satisfy that $k + 2n \in J$ if $k \in J$. This yields $k \in I_{\ell+1}$ if $k \in I_\ell$. Again the number of indices is not changed by this map. Accordingly the resulting tuple $I = (I_\ell)_{\ell=1}^N$ is included in the claimed set. This proves the bijection since both maps are inverse to each other. □

EXAMPLE 5.23. Let $n = 3$ and $N = 2$. In the approximation $\mathrm{Gr}_2^a(\widehat{\mathfrak{gl}}_3)$ we have a cell which parametrised by the pair of index sets

$$I = \big(I_1 = \{1,4\}, I_2 = \{1,3,4,6\}\big).$$

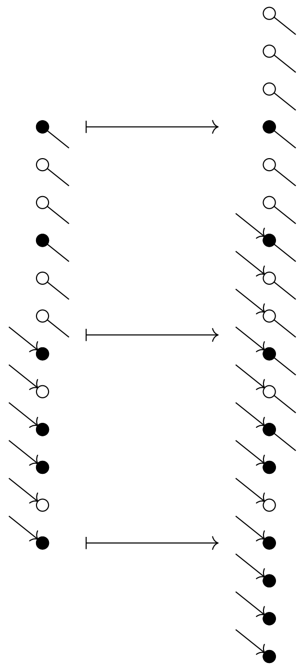
The corresponding index set $J \in \mathcal{I}_6(12)$ is given by

$$J = \{1,4,7,9,10,12\} = \{1,4,1+6,3+6,4+6,6+6\}.$$

The index set $\tilde{J} \in \mathcal{I}_9(18)$ which is computed using the map above is given by

$$\tilde{J} = \{4,7,10,12,13,15,16,17,18\} = \{1+3,4+3,7+3,9+3,10+3,12+3\} \cup \{16,17,18\}.$$

For the successor closed subquivers, the mapping is of the form



5.5.2. Ind-Variety Structure of the Affine Grassmannian. In this section we use the properties of the approximations of the Feigin-degenerate affine Grassmannian and the maps between them to introduce the maps for the ind-variety structure of the non-degenerate affine Grassmannian.

For this purpose we need an alternative description of the cells in the quiver Grassmannian arising from the successor closed subquivers in the coefficient quiver of $A_{2N} \otimes \mathbb{C}^n$. Analogous to approximations of the Feigin degeneration we prove the following statement.

PROPOSITION 5.24. The cells of the quiver Grassmannian

$$\mathrm{Gr}_{Nn}^{A_{2N}}\big(A_{2N} \otimes \mathbb{C}^n\big)$$

are in bijection with the set

$$\mathcal{I}_{nN}^0(2nN) := \Big\{ I \subset [2nN] : |I| = nN \text{ and } k+n \in I \text{ if } k \in I \Big\}.$$

Again there exists an alternative characterisation using tuples of index sets. For the proof of the subsequent proposition we have to replace N by $2N$ and $2n$ by n in the proof of the analogous statement for the approximations of the Feigin degenerations.

PROPOSITION 5.25. The cells of the quiver Grassmannian

$$\mathrm{Gr}_{nN}^{A_{2N}}\left(A_{2N}\otimes\mathbb{C}^n\right)$$

are parametrised by the set

$$\mathcal{I}_{2N,nN}(n) := \Big\{ I = (I_\ell)_{\ell\in[2N]} : I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{2N} \subseteq [n] \text{ and } \sum_{\ell=1}^{2N} |I_\ell| = nN \Big\}.$$

We can use the same map Ψ_N as above to introduce the ind-variety structure of the non-degenerate affine Grassmannian. The proof that the image of a point p in the quiver Grassmannian

$$\mathrm{Gr}_{nN}^{A_{2N}}\left(A_{2N}\otimes\mathbb{C}^n\right)$$

is a point $\tilde{p}$ in the quiver Grassmannian

$$\mathrm{Gr}_{n(N+1)}^{A_{2(N+1)}}\left(A_{2(N+1)}\otimes\mathbb{C}^n\right)$$

is similar to the proof for the approximations of the Feigin degeneration. We describe the points explicitly as the span of certain w_s which have the same shape as above. Only the parametrising index sets for the cells have changed. All steps in the proof work in the same way. Hence we arrive at an closed embedding

$$\Phi_N : \mathrm{Gr}_{nN}^{A_{2N}}\left(A_{2N}\otimes\mathbb{C}^n\right) \to \mathrm{Gr}_{n(N+1)}^{A_{2(N+1)}}\left(A_{2(N+1)}\otimes\mathbb{C}^n\right)$$

which preserves the dimension of the cells.

REMARK. For the parametrisation of the cells by the index tuples in $\mathcal{I}_{2N,nN}(n)$ the map between the approximations has a simpler description than for $\mathcal{I}_{nN}^0(2nN)$. Namely let I be an index tuple in $\mathcal{I}_{2N,nN}(n)$ then the image of I under the map Φ_N is given by the tuple $\tilde{I} \in \mathcal{I}_{2N+2,n(N+1)}(n)$ where

$$\tilde{I}_1 = \emptyset, \tilde{I}_{\ell+1} = I_\ell \text{ for } \ell \in [2N] \text{ and } \tilde{I}_{nN+2} = [n].$$

On the level of cells we study the maps between the approximations for the different parametrisations in the subsequent example.

EXAMPLE 5.26. For $n = 5$ and $N = 1$, the pair of index sets

$$I = \big(I_1 = \{1,4\}, I_2 = \{1,2,4\}\big)$$

which is included in $\mathcal{I}_{2,5}(5)$ describes a cell in the approximation $\mathrm{Gr}_1(\widehat{\mathfrak{gl}}_5)$. Its image in $\mathcal{I}_{4,10}(5)$ under the map Φ_N is given by

$$\tilde{I} = \big(I_1 = \emptyset, I_2 = \{1,4\}, I_3 = \{1,2,4\}, I_4 = [n]\big).$$

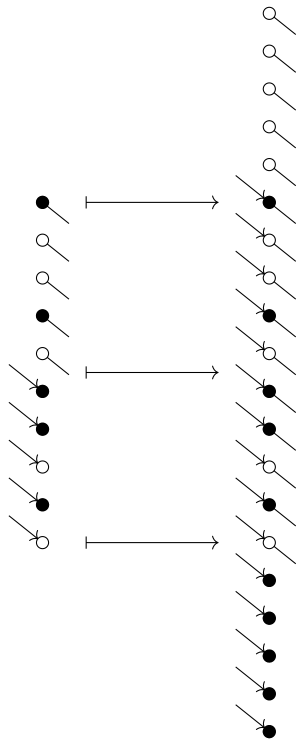
In the parametrisation by one index set, I corresponds to

$$J = \{1,4,6,7,9\} = \{1,4,1+5,2+5,4+5\} \in \mathcal{I}_5^0(10)$$

and the image of J in $\mathcal{I}_{10}^0(20)$ is given by

$$\begin{aligned}\tilde{J} &= \{6,9,11,12,14,16,17,18,19,20\}\\ &= \{1+5,4+5,6+5,7+5,9+5\} \cup \{16,17,18,19,20\}.\end{aligned}$$

For the successor closed subquiver interpretation the map is given by



5.5.3. Ind-Variety Structure of Partial Degenerations of the Affine Grassmannian. The parametrisations of the cells in the approximations of the affine Grassmannian and the Feigin-degenerate affine Grassmannian using index sets or tuples of index sets as studied above are more complicated for the intermediate degenerations. This is the case because in the coefficient quiver of the representation

$$M_N^k := A_N \otimes \mathbb{C}^{2k} \oplus A_{2N} \otimes \mathbb{C}^{n-k}$$

the segments do not all have the same length.

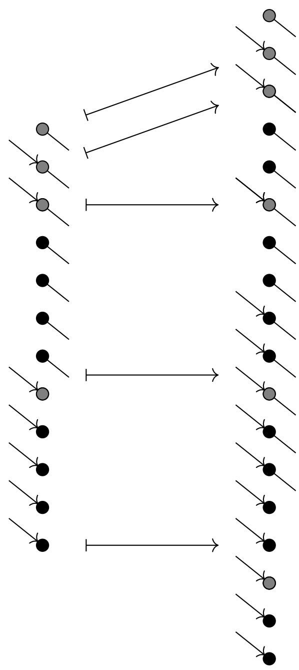
Hence it is not possible to describe the maps between the approximations via the index shift Ψ_N which was used for the Feigin degeneration and the non-degenerate affine Grassmannian. Nevertheless there exists a universal approach to compute the assignment of basis vectors in all approximations. It specialises to the cases studied above.

For the definition of this map we choose the successor closed subquiver parametrisation of the cells in order to visualise where the difference in the assignment arises. In both cases of the map between the approximations that we have studied so far the number of points in the coefficient quiver grows by $2n$. We add n points below and n points above the points of M_N^k. Then we extend the segments of M_N^k to the new points matching the structure of M_{N+1}^k. This is the step where the intermediate degenerations behave different from the Feigin degeneration and the non-degenerate affine Grassmannian.

EXAMPLE 5.27. We study the map between the coefficient quivers of M_N^k and $M_{N+1}^r k$ for $n = 3$, $N = 2$ and $k = 2$. The map between the coefficient quivers of

$$M_2^2 := A_2 \otimes \mathbb{C}^4 \oplus A_4 \otimes \mathbb{C}^1 \text{ and } M_3^2 := A_3 \otimes \mathbb{C}^4 \oplus A_6 \otimes \mathbb{C}^1$$

is given by



where we coloured the vertices corresponding to the long segment in gray.

The part of the coefficient quiver of M_N^k corresponding to the segments of $A_N \otimes \mathbb{C}^{n+k}$ is embedded into the coefficient quiver of M_{N+1}^k. The long segment and the two upper short segments are extended to the three new vertices below the embedded quiver. This is analogous to the previously studied embeddings.

But on top of the embedded quiver we have to change the assignment in order to match the structure of M_{N+1}^k. Namely the other two short segments have to be extended here such that their starting points are below the other vertices corresponding to the long segment which is also extended by one point at the top.

In general the image of the coefficient quiver of $A_N \otimes \mathbb{C}^{n+k}$ inside the coefficient quiver of M_{N+1}^k determines how the new arrows of this coefficient quiver have to be drawn. Only for the two special cases studied above they have the nice interpretation in terms of the index shift. For the intermediate degeneration the index shift applies only to points in cells whose corresponding subquivers live completely inside of the coefficient quiver of $A_N \otimes \mathbb{C}^{n+k}$. For the other cells it is more complicated to describe the mapping of the index sets describing the cells.

Based on the method described above it is possible to define polynomial maps

$$\Phi_N^k : \mathrm{Gr}_{Nn}^{A_{2N}}\big(A_N \otimes \mathbb{C}^{2k} \oplus A_{2N} \otimes \mathbb{C}^{n-k}\big) \to \mathrm{Gr}_{(N+1)n}^{A_{2N+2}}\big(A_{N+1} \otimes \mathbb{C}^{2k} \oplus A_{2N+2} \otimes \mathbb{C}^{n-k}\big)$$

for every $k \in \{0, 1, 2, \dots, n\}$ such that the image in the bigger approximation is closed.

Moreover this map sends cells to cells and it preserves the dimension of the cells if the cell we start with is representable by a subquiver in the coefficient quiver of $A_N \otimes \mathbb{C}^{n+k}$. This follows with the same arguments as for the Feigin degeneration of the affine Grassmannian.

5.6. The Action of the Automorphism Groups in the Limit

In this section we examine the action of the automorphism groups $\mathrm{Aut}_{\Delta_1}(M_N^k)$ on the quiver Grassmannians

$$\mathrm{Gr}_{Nn}^{A_N}\big(M_N^k\big)$$

for the special case $k \in \{0, n\}$ and if there exists an embedding of automorphism groups

$$\varphi_N^k : \mathrm{Aut}_{\Delta_1}(M_N^k) \to \mathrm{Aut}_{\Delta_1}(M_{N+1}^k)$$

which is compatible with the maps

$$\Phi_N^k : \mathrm{Gr}_{Nn}^{A_{2N}}\big(M_N^k\big) \to \mathrm{Gr}_{(N+1)n}^{A_{2N+2}}\big(M_{N+1}^k\big).$$

Before we study the the automorphism groups $\mathrm{Aut}_{\Delta_1}(M_N)$ and $\mathrm{Aut}_{\Delta_1}(M_N^a)$ we work out the explicit shape of the endomorphisms

$$\mathrm{End}_{\Delta_1}\big(A_N\big) = \mathrm{Hom}_{\Delta_1}\big(A_N, A_N\big)$$

for the indecomposable nilpotent representation A_N of the loop quiver.

This quiver representation is of the form

$$A_N = \Big(\mathbb{C}^N, s_1\Big)$$

where the map $s_1 : \mathbb{C}^N \to \mathbb{C}^N$ acts on the basis vectors e_i as

$$s_1 e_i = \begin{cases} e_{i+1} & \text{if } i < N \\ 0 & \text{otherwise.} \end{cases}$$

For a vector $v \in \mathbb{C}^N$ this corresponds to left multiplication with the matrix $H_1 \in M_N(\mathbb{C})$ with entries $h_{i,j} := \delta_{i-1,j}$ for $i, j \in [N]$. The endomorphisms of the representation A_N are all linear maps $\rho : \mathbb{C}^N \to \mathbb{C}^N$ such that

$$\rho \circ H_1 = H_1 \circ \rho.$$

The entries of $\rho \circ H_1$ are given by

$$\big(\rho \circ H_1\big)_{i,j} = \sum_{k=1}^{N} \rho_{i,k} H_{k,j} = \sum_{k=1}^{N} \rho_{i,k}\delta_{k-1,j} = \begin{cases} \rho_{i,j+1} & \text{if } j < N \\ 0 & \text{otherwise.} \end{cases}$$

For $H_1 \circ \rho$ we obtain the entries

$$\big(H_1 \circ \rho\big)_{i,j} = \sum_{k=1}^{N} H_{i,k}\rho_{k,j} = \sum_{k=1}^{N} \delta_{i-1,k}\rho_{k,j} = \begin{cases} \rho_{i-1,j} & \text{if } i > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Accordingly we obtain $\rho_{i-1,j} = \rho_{i,j+1}$ for $i > 1$ and $j < N$ which is equivalent to $\rho_{i,j} = \rho_{i+1,j+1}$ for $i, j < N$. Moreover we have $\rho_{1,j} = 0$ for $j > 1$ and $\rho_{i,n} = 0$ for $i < N$. This yields that all entries of ρ above the main diagonal are equal to zero and that the entries are constant on the remaining N diagonals.

Hence the matrix ρ is described completely by the values of the entries $\rho_{i,1}$ for $i \in [N]$ and these entries are independent of each other. This implies that the space of endomorphisms of A_N is N-dimensional. The automorphisms of A_N are the endomorphisms such that $\rho_{1,1}$ is invertible since this implies that the corresponding matrix ρ is invertible.

Given a tuple $(\lambda_k)_{k\in[N]}$ with entries $\lambda_k \in \mathbb{C}$ we define the entries of the lower triangular matrix $A(\lambda) \in M_N(\mathbb{C})$ by

$$a_{i,j} := \begin{cases} \lambda_k & \text{if } k = i - j + 1 \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is an element of the automorphism group $\mathrm{Aut}_{\Delta_1}(A_N)$ if $\lambda_1 \neq 0$ and all elements of the automorphism group can be described in this way. For this reason the automorphism group is also N-dimensional.

It acts on the quiver Grassmannian by multiplication from the left with the vectors spanning a point in the quiver Grassmannian. This description of the points in the quiver Grassmannians is introduced in the section about the ind-variety structure and the image of the action is independent of the choice of vectors spanning the point.

The quiver representation $M_N = A_{2N} \otimes \mathbb{C}^n$ corresponds to the map

$$s_n : \mathbb{C}^{2nN} \to \mathbb{C}^{2nN}$$

and analogous we compute the endomorphisms of M_N as the matrices in $M_{2nN}(\mathbb{C})$ which commute with the matrix $H_n \in M_{2nN}(\mathbb{C})$. These matrices can be described as follows. For a tuple

$$\lambda := \big(\lambda_k^{(i,j)}\big) \text{ with } k \in [2N] \text{ and } i,j \in [n]$$

let $M_k(\lambda)$ be the matrix with entries $\lambda_k^{(i,j)}$ for $i,j \in [n]$. Define the $2N \times 2N$ block matrix $A_\lambda \in M_{2nN}(\mathbb{C})$ with the blocks

$$A_{p,q} := \begin{cases} M_k(\lambda) & \text{if } k = p - q + 1 \\ \mathbf{0}_{n,n} & \text{otherwise.} \end{cases}$$

Independent of the choice of λ this describes an endomorphism of M_N and all endomorphisms are obtained in this way. If we additionally require that the matrix $M_1(\lambda) \in M_n(\mathbb{C})$ is invertible it is sufficient to describe the automorphisms of M_N. Hence the group $\mathrm{Aut}_{\Delta_1}(M_N)$ is $2Nn^2$-dimensional.

Based on this description of the automorphism groups we define the embedding

$$\varphi_N : \mathrm{Aut}_{\Delta_1}(M_N) \to \mathrm{Aut}_{\Delta_1}(M_{N+1})$$

by $\varphi_N(A_\lambda) := A_{\hat{\lambda}}$ where $\hat{\lambda}$ is obtained from λ as

$$\hat{\lambda}_k^{(i,j)} := \begin{cases} \lambda_k^{(i,j)} & \text{if } k < 2N \\ 0 & \text{otherwise.} \end{cases}$$

This embedding is compatible with the actions of the automorphism groups on the quiver Grassmannians.

Proposition 5.28. Let $A \in \mathrm{Aut}_{\Delta_1}(M_N)$ be an automorphism of the quiver Grassmannian

$$\mathrm{Gr}_{Nn}^{A_{2N}}\big(M_N\big) \cong \mathrm{Gr}_N\big(\,\widehat{\mathfrak{gl}}_n\,\big).$$

Then the diagram

$$\begin{array}{ccc} \mathrm{Gr}_N\big(\,\widehat{\mathfrak{gl}}_n\,\big) & \xrightarrow{\quad A \quad} & \mathrm{Gr}_N\big(\,\widehat{\mathfrak{gl}}_n\,\big) \\ \Phi_N \big\downarrow & /\!/\!/ & \big\downarrow \Phi_N \\ \mathrm{Gr}_{N+1}\big(\,\widehat{\mathfrak{gl}}_n\,\big) & \xrightarrow[\varphi_N(A)]{} & \mathrm{Gr}_{N+1}\big(\,\widehat{\mathfrak{gl}}_n\,\big) \end{array}$$

commutes.

PROOF. From the section about the ind-variety structure of the affine Grassmannian we know that the points p in the approximations are described as

$$p = \mathrm{Span}\{w_1, \dots, w_{nN}\}$$

where

$$w_t = v_{k_t} + \sum_{j>k_t, j\notin I} \lambda_{j,t} v_j$$

and $I \in \mathcal{I}^0_{nN}(2nN)$ is a index set parametrising some fixed point. In this parametrisation the action of an element A of the automorphism group of the representation M_N on the point p is defined as $A.p := \mathrm{Span}\{Aw_1, \dots, Aw_{nN}\}$. The image of the point p under the map

$$\Phi_N : \mathrm{Gr}_N\big(\,\widehat{\mathfrak{gl}}_n\,\big) \to \mathrm{Gr}_{N+1}\big(\,\widehat{\mathfrak{gl}}_n\,\big)$$

is computed as

$$\Phi_N(p) = \mathrm{Span}\Big(\big\{s_n w_t : t \in [nN]\big\} \cup \big\{v_{2nN+n+i} : i \in [n]\big\}\Big).$$

On the vectors

$$\Big\{v_{2nN+i} : i \in [n]\Big\}$$

an element $A_\lambda \in \mathrm{Aut}_{\Delta_1}(M_N)$ acts with the block $M_1(\lambda)$ which is invertible. Hence we obtain

$$\mathrm{Span}\big\{v_{2nN+n+i} : i \in [n]\big\} = \mathrm{Span}\big\{A_\lambda v_{2nN+n+i} : i \in [n]\big\}.$$

On a basis vector $v_{nk+j} \in \mathbb{C}^{2nN}$ the automorphism A_λ acts with the blocks $M_\ell(\lambda)$ for $\ell \in [2N-k+1]$. On the image $s_n v_{nk+j} = v_{nk+j+n} \in \mathbb{C}^{2n(N+1)}$ the automorphism $\varphi_N(A_\lambda)$ acts with the blocks $M_\ell(\lambda)$ for $\ell \in [2N+2-k]$. This means that there is an additional action of the block $M_{2N-k+2}(\lambda)$. The image of this additional action lives on the new basis vectors v_j with $j > 2nN+n$. For the other blocks the action on the shift of w_t lives over the shift of the basis vectors it was living over for the unshifted version of w_j, i.e.

$$Aw_t = s_{-n}\phi_N(A)s_n w_t.$$

It is not important what happens over the basis vectors v_j with $j > 2nN+n$ because these basis vectors are all included in the generating set of the span and changes of these entries for the other generators do not affect the span of all generators. Accordingly we obtain

$$\Phi_N(A.p) = \varphi_N(A).\Phi_N(p).$$

□

Now we want to study the case $k = n$ where we have automorphisms of the representation

$$M^a_N = A_N \otimes \mathbb{C}^{2n}$$

which corresponds to the map

$$s_{2n} : \mathbb{C}^{2nN} \to \mathbb{C}^{2nN}.$$

Based on the description of the automorphisms of A_N we compute that the elements of the group $\mathrm{Aut}_{\Delta_1}(M^a_N)$ are parametrised as follows. Let μ be the tuple

$$\mu := \big(\mu^{(i,j)}\big) \text{ with } k \in [N] \text{ and } i,j \in [2n]$$

and $M_k(\mu)$ be the matrix with entries $\mu_k^{(i,j)}$ for $i, j \in [2n]$. Define the $N \times N$ block matrix $A_\mu \in M_{2nN}(\mathbb{C})$ with the blocks

$$A_{p,q} := \begin{cases} M_k(\mu) & \text{if } k = p - q + 1 \\ \mathbf{0}_{2n,2n} & \text{otherwise.} \end{cases}$$

The matrix A_μ is an element of the automorphism group $\mathrm{Aut}_{\Delta_1}(M_N^a)$ if and only if $\det A_{1,1} \neq 0$ and this group is $4Nn^2$-dimensional. We define the embedding

$$\varphi_N^a : \mathrm{Aut}_{\Delta_1}(M_N^a) \to \mathrm{Aut}_{\Delta_1}(M_{N+1}^a)$$

by $\varphi_N^a(A_\mu) := A_{\hat{\mu}}$ where $\hat{\mu}$ is obtained from μ as

$$\hat{\mu}_k^{(i,j)} := \begin{cases} \mu_k^{(i,j)} & \text{if } k < N \\ 0 & \text{otherwise.} \end{cases}$$

Remark. This embedding is not compatible with the map

$$\Phi_N^a : \mathrm{Gr}_N^a\big(\widehat{\mathfrak{gl}}_n\big) \to \mathrm{Gr}_{N+1}^a\big(\widehat{\mathfrak{gl}}_n\big)$$

because for Φ_N^a we have an index shift of the vectors spanning the points in the Grassmannians by n but the blocks in the elements of the automorphism group have size $2n$.

Moreover it is not possible to define an embedding of the automorphism groups such that the action is equivariant for the map Φ_N^a. We would have to split the blocks of size $2n$ into smaller blocks of size n to be compatible with the index shit by n. This shift would require a flip of the $n \times n$ subblocks on the diagonal of the $2n \times 2n$ blocks in order to have the same blocks acting on a vector before and after the embedding Φ_N^a. But the upper right $n \times n$ subblock of $M_1(\mu)$ cannot be embedded into an automorphism of M_{N+1}^a in order to match its action on M_N^a. It would have to be located in a block above the diagonal and this is not possible since these blocks are zero for elements of the automorphism group $\mathrm{Aut}_{\Delta_1}(M_{N+1}^a)$.

If we apply $\Phi_{N+1}^a \circ \Phi_N^a$ we have an index shift of $2n$ and this is compatible with the embedding $\varphi_{N+1}^a \circ \varphi_N^a$ of the automorphism group. It is checked in the same way as for the non-degenerate affine Grassmannian that the square in the subsequent proposition commutes.

Proposition 5.29. Let $A \in \mathrm{Aut}_{\Delta_1}(M_N^a)$ be an automorphism of the quiver Grassmannian

$$\mathrm{Gr}_{Nn}^{A_N}(M_N^a) \cong \mathrm{Gr}_N^a\big(\widehat{\mathfrak{gl}}_n\big).$$

Then the diagram

$$\begin{array}{ccc} \mathrm{Gr}_N^a\big(\widehat{\mathfrak{gl}}_n\big) & \xrightarrow{\quad A \quad} & \mathrm{Gr}_N^a\big(\widehat{\mathfrak{gl}}_n\big) \\ {\scriptstyle \Phi_{N+1}^a \circ \Phi_N^a}\downarrow & /\!\!/ & \downarrow{\scriptstyle \Phi_{N+1}^a \circ \Phi_N^a} \\ \mathrm{Gr}_{N+2}^a\big(\widehat{\mathfrak{gl}}_n\big) & \xrightarrow[\varphi_{N+1}^a \circ \varphi_N^a(A)]{} & \mathrm{Gr}_{N+2}^a\big(\widehat{\mathfrak{gl}}_n\big) \end{array}$$

commutes.

5.7. Geometric Properties

In this section we examine the geometric properties of the partial degenerations of the affine Grassmannian and their approximations which arise from the study of the corresponding quiver Grassmannians.

The subsequent corollary is an immediate consequence of the results concerning the quiver Grassmannians for the loop quiver as studied in the first section of this chapter.

COROLLARY 5.30. $\mathrm{Gr}_N^a(\widehat{\mathfrak{gl}}_n)$ is an projective variety of dimension Nn^2. It is irreducible, normal, Cohen-Macaulay and has rational singularities.

PROOF. The first part of the statement follows from Proposition 5.2 by setting $x = y = n$. Irreducibility and the geometric properties are obtained from Proposition 5.3. □

For the non-degenerate affine Grassmannian there is an analogous result.

COROLLARY 5.31. $\mathrm{Gr}_N(\widehat{\mathfrak{gl}}_n)$ is an projective variety of dimension $2N\lfloor n/2\rfloor\lceil n/2\rceil$. It is irreducible, normal, Cohen-Macaulay and has rational singularities.

PROOF. By Lemma 5.16 we know that the finite approximations of the affine Grassmannian are given by

$$\mathrm{Gr}_{Nn}^{A_{2N}}\big(A_{2N}\otimes\mathbb{C}^n\big).$$

For even n we can apply Lemma 3.22 to compute the dimension of the approximations. We set

$$X_{2N} := Y_{2N} := A_{2N}\otimes\mathbb{C}^{n/2}$$

and obtain

$$\dim \mathrm{Gr}_{Nn}^{A_{2N}}\big(A_{2N}\otimes\mathbb{C}^n\big) = 2N(n-n/2)n/2$$

which matches the claimed formula since n is even. The irreducible components are parametrised by the set described in Lemma 3.23 which contains only one element since the loop quiver has only one vertex. The geometric properties are again obtained from Proposition 5.3.

For odd n we can apply Theorem 2.3 since A_{2N} is a bounded injective representation of the loop quiver. This allows us to study orbits in the variety of quiver representations in order to find the irreducible components of the quiver Grassmannian providing the approximation. As in the proof of Proposition 3.21 we know that increasing the length of the words parametrising an orbit also increases the dimension of the orbit. Since we only have one letter, it is possible to glue all words as long as the length of the glued word is not longer than $2N$. Hence

$$U := A_{2N}\otimes\mathbb{C}^{\lfloor n/2\rfloor}\oplus A_N$$

is a representative of the highest dimensional orbit in the bounded variety of quiver representations and all other orbits have strictly smaller dimension. It remains to compute the dimension of the stratum of U. By Lemma 1.3 we obtain

$$\begin{aligned}\dim \mathrm{Gr}_{Nn}^{A_{2N}}(M) &= \dim \mathrm{Hom}_Q(U,M) - \dim \mathrm{End}_Q(U)\\ &= 2N\lfloor n/2\rfloor n + Nn - \big(2N\lfloor n/2\rfloor\lfloor n/2\rfloor + 2N\lfloor n/2\rfloor + N\big)\\ &= 2N\lfloor n/2\rfloor\big(n-\lfloor n/2\rfloor\big) + Nn - \big(2N\lfloor n/2\rfloor + N\big)\\ &= 2N\lfloor n/2\rfloor\lceil n/2\rceil + Nn - \big(N(n-1)+N\big)\\ &= 2N\lfloor n/2\rfloor\lceil n/2\rceil.\end{aligned}$$

□

Remark. For the Feigin-degenerate affine Grassmannian and the non-degenerate affine Grassmannian we have show in Section 5.5 that the closed embeddings of the finite approximations preserve the cell structure and additionally the dimension of the cells. Hence the ind-topology and the Zariski topology on $\mathrm{Gr}^a(\widehat{\mathfrak{gl}}_n)$ coincide by a result of I. Stampfli [**71**].

Conjecture 5.32. *The approximation to the parameter N of the k-linear degenerate affine Grassmannian is a projective variety of dimension*

$$\dim \mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n) = N\Big(nk + 2\Big(n-k-\Big\lfloor\frac{n-k}{2}\Big\rfloor\Big)\Big\lfloor\frac{n-k}{2}\Big\rfloor\Big).$$

It is irreducible if $k\in\{0,n\}$ or $n-k$ is even. If $n-k$ is odd it has $N+1$ irreducible components.

The proof of this statement requires a completely different approach since the direct summands of the quiver representation used for the approximation do not have the same length and hence can not be all bounded injective representations. This means that looking at the variety of quiver representations does not help to find the irreducible components of the quiver Grassmannian since Theorem 2.3 does not hold in this setting.

Hence there is no closure preserving bijection between orbits in the variety of quiver representations and strata in the quiver Grassmannian. So far we have no methods to compute tight upper bounds for the dimension of the strata in the quiver Grassmannian directly without using the inclusion relations of the orbits in the variety of quiver representations.

For all computations in the appendix, the above formulas give exactly the dimension of the Grassmannian and the number of irreducible components. Based on the methods introduced in this thesis we are only able to show that this formula gives a lower bound for the dimension of the finite approximations.

Proposition 5.33. The approximation to the parameter N of the k-linear degenerate affine Grassmannian is a projective variety of dimension

$$\dim \mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n) \geq N\Big(nk + 2\Big(n-k-\Big\lfloor\frac{n-k}{2}\Big\rfloor\Big)\Big\lfloor\frac{n-k}{2}\Big\rfloor\Big).$$

Proof. By Lemma 5.16 the finite approximations are isomorphic to the quiver Grassmannians

$$\mathrm{Gr}_{Nn}^{A_{2N}}(M_k)$$

where

$$M_k := A_{2N}\otimes\mathbb{C}^{n-k}\oplus A_N\otimes\mathbb{C}^{2k}.$$

For the computation of the lower bound on the dimension of these quiver Grassmannians take the subrepresentation

$$U_k := A_{2N} \otimes \mathbb{C}^{\lfloor \frac{n-k}{2} \rfloor} \oplus A_N \otimes \mathbb{C}^{n-2\lfloor \frac{n-k}{2} \rfloor}$$

and compute

$$\dim \mathrm{Gr}_{Nn}^{A_{2N}}\big(M_k\big) \geq \dim \mathrm{Hom}_Q\big(U_k, M_k\big) - \dim \mathrm{End}_Q\big(U_k\big).$$

The space of homomorphisms has dimension

$$\begin{aligned}
\dim \mathrm{Hom}_Q\big(U_k, M_k\big) &= 2N\Big\lfloor \frac{n-k}{2} \Big\rfloor \big(n-k\big) + N\Big\lfloor \frac{n-k}{2} \Big\rfloor 2k \\
&\quad + N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)\big(n-k\big) + N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)2k \\
&= 2N\Big\lfloor \frac{n-k}{2} \Big\rfloor n + N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)\big(n-k\big) \\
&\quad + 2N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)k
\end{aligned}$$

and the space of endomorphisms has dimension

$$\begin{aligned}
\dim \mathrm{End}_Q\big(U_k\big) &= 2N\Big\lfloor \frac{n-k}{2} \Big\rfloor \Big\lfloor \frac{n-k}{2} \Big\rfloor + N\Big\lfloor \frac{n-k}{2} \Big\rfloor \Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big) \\
&\quad + N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)\Big\lfloor \frac{n-k}{2} \Big\rfloor + N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)^2 \\
&= 2N\Big\lfloor \frac{n-k}{2} \Big\rfloor \Big(n - \Big\lfloor \frac{n-k}{2} \Big\rfloor\Big) + N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)^2.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\dim \mathrm{Gr}_{Nn}^{A_{2N}}\big(M_k\big) &\geq \dim \mathrm{Hom}_Q\big(U_k, M_k\big) - \dim \mathrm{End}_Q\big(U_k\big) \\
&= 2N\Big\lfloor \frac{n-k}{2} \Big\rfloor n + N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)\big(n-k\big) \\
&\quad + 2N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)k \\
&\quad - 2N\Big\lfloor \frac{n-k}{2} \Big\rfloor \Big(n - \Big\lfloor \frac{n-k}{2} \Big\rfloor\Big) - N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)^2 \\
&= 2N\Big\lfloor \frac{n-k}{2} \Big\rfloor \Big\lfloor \frac{n-k}{2} \Big\rfloor + N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)\big(n-k\big) \\
&\quad + 2N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)k - N\Big(n - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)^2 \\
&= N\Big(2\Big\lfloor \frac{n-k}{2} \Big\rfloor^2 + n^2 - nk - 2n\Big\lfloor \frac{n-k}{2} \Big\rfloor + 2k\Big\lfloor \frac{n-k}{2} \Big\rfloor + 2nk \\
&\quad - 4k\Big\lfloor \frac{n-k}{2} \Big\rfloor - n^2 - 4\Big\lfloor \frac{n-k}{2} \Big\rfloor^2 + 4n\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big) \\
&= N\Big(nk + 2\big(n-k\big)\Big\lfloor \frac{n-k}{2} \Big\rfloor - 2\Big\lfloor \frac{n-k}{2} \Big\rfloor^2\Big) \\
&= N\Big(nk + 2\Big(n - k - \Big\lfloor \frac{n-k}{2} \Big\rfloor\Big)\Big\lfloor \frac{n-k}{2} \Big\rfloor\Big).
\end{aligned}$$

□

REMARK. For $k = 0$ and $k = n$ we rediscover the formulas for the dimension of the approximations as computed above. This suggests that the bound is also sharp for the approximations of the intermediate degenerations of the affine Grassmannian. Moreover the bound is sharp in all examples which were checked using the computer program.

For the proof that equality holds we need different methods to compute the strata of highest dimension. A first step in this direction is the generalisation of the result about the cellular decomposition of the quiver Grassmannians to the class of Grassmannians which is needed for the approximations of the intermediate degenerations.

PROPOSITION 5.34. For every $L \in \mathrm{Gr}^{A_N}_{nN}(M^k_N)^T$, the subset $\mathcal{C}(L) \subseteq \mathrm{Gr}^{A_N}_{nN}(M^k_N)$ is an affine space and the quiver Grassmannian admits a cellular decomposition

$$\mathrm{Gr}^{A_N}_{nN}(M^k_N) = \coprod_{L \in \mathrm{Gr}^{A_N}_{nN}(M^k_N)^T} \mathcal{C}(L).$$

For $k = 0$ and $k = n$ this result follows from Proposition 5.6. But for the intermediate degenerations not all summands of M^k_N are of the same length such that the torus action on the quiver Grassmannian has to be defined in a different way.

PROOF. We arrange the segments in the coefficient quiver of M^k_N as in Section 5.5. The assumption $d(\alpha) := n + k$ induces a grading of the vertices in the coefficient quiver which satisfies the assumption of Theorem 4.7. Hence by Proposition 4.9 the number successor closed subquivers on nN vertices in the coefficient quiver of M^k_N equals the Euler Poincaré characteristic of these quiver Grassmannians.

Analogous to the proof of Theorem 4.10 we show that the attracting sets of the torus fixed points are affine spaces and describe a cellular decomposition of the quiver Grassmannian. □

Using the shape of the coefficient quiver of M^k_N and the fact that the strata in the quiver Grassmannians are parametrised by certain successor closed subquivers therein we might help prove that the lower bound on the dimension of the quiver Grassmannians as introduced above is also an upper bound. We need a construction proving that there can not be successor closed subquivers with more than the desired amount of holes below the starting points. This could be done by looking at the inclusion relations of the cells. But these relations are more complicated as in the case where all indecomposable summands have the same length and the proof for the upper bound will not be part of this thesis.

5.8. Cellular Decomposition and Poincaré Series

In this section we study the Euler Poincaré characteristic of the approximations and the Poincaré series of the partial degenerate affine Grassmannians. For the Feigin-degenerate affine Grassmannian we have the following description of the Euler Poincaré characteristic of the approximations.

LEMMA 5.35. *The Euler Poincaré characteristic $\chi_{n,N}$ of the finite approximation $\mathrm{Gr}^a_N(\widehat{\mathfrak{gl}}_n)$ is given by*

$$\chi_{n,N} = \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \binom{(n-k)(N+1)+n-1}{2n-1}.$$

Proof. The finite approximations of the Feigin-degenerate affine Grassmannian are given by the quiver Grassmannians $\mathrm{Gr}_{Nn}(A_N \otimes \mathbb{C}^{2n})$ for the loop quiver. The Euler Poincaré characteristic of this quiver Grassmannian equals the number of fixed points of the $\mathbb{C}^*$-action on this quiver Grassmannian.

The torus fixed points are in bijection with successor closed subquivers in the coefficient quiver of $A_N \otimes \mathbb{C}^{2n}$ with Nn many marked points. These subquivers are parametrised by the length of the segments embedded into the $2n$ copies of A_N, i.e.

$$\chi_{n,N} := \Big|\mathcal{C}^{(2n)}_{nN}(\Delta_1, \mathrm{I}_N)\Big| = \Big|\Big\{\mathbf{p} \in \mathbb{Z}^{2n} : 0 \le p_i \le N, \sum_{i\in[2n]} p_i = Nn\Big\}\Big|.$$

where the description of the set of cells was obtained in the first section of this chapter. Setting $a_i = p_i$ and $b_i = N - p_{n+i}$ for $i \in [n]$ we obtain the equality

$$\chi_{n,N} = \Big|\Big\{(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^n \times \mathbb{Z}^n : 0 \le a_i, b_i \le N, \sum_{i\in[n]} a_i = \sum_{i\in[n]} b_i\Big\}\Big|.$$

The cardinality of this set is given in the article of M. Nathanson [**61**, p. 8]. □

Remark. For the intermediate degenerations of the affine Grassmannian it is more complicated to find a formula for the Euler Poincaré characteristic of the finite approximations since the length of the segments parametrising a cell is not bounded by one parameter but two.

The maps introducing an ind-variety structure on the approximations by quiver Grassmannians allow us to compute the limit of the Poincaré polynomials for the approximations. First we want to compute this limit for the Feigin degeneration and then we study it for the partial degenerations where the proof works analogous to the method developed in the Feigin setting.

Theorem 5.36. *The limit of the Poincaré polynomials $p_{n,N}(q)$ of the finite approximations $\mathrm{Gr}^a_N(\widehat{\mathfrak{gl}}_n)$ is given by*

$$\lim_{N\to\infty} p_{n,N}(q) = p_n(q) := \prod_{k=1}^{2n-1} (1 - q^k)^{-1}.$$

The prove of this theorem is based on the observation that the number of k-dimensional cells stabilises for N big enough.

Proposition 5.37. Let $b_k^{(N)}$ be the number of k-dimensional cells in the finite approximation $\mathrm{Gr}^a_N(\widehat{\mathfrak{gl}}_n)$. Then

(i) $b_k^{(N)} \le b_k^{(N+1)}$ for all $N \ge 1$ and

(ii) $b_k^{(N)} = b_k^{(N_k)}$ for all $N \ge N_k$ where $N_k := \lceil k/n \rceil$.

Proof. We start with the first part. In the previous section we have seen that the map Φ_N between the finite approximations preserves the dimensions of the cells. In Proposition 5.21 we have shown that the image of a k-dimensional cell in $\mathrm{Gr}^a_N(\widehat{\mathfrak{gl}}_n)$ is a k-dimensional cell in $\mathrm{Gr}^a_{N+1}(\widehat{\mathfrak{gl}}_n)$. Thus the number of k-dimensional cells in the bigger approximation can not be smaller than the number of k-dimensional cells in the smaller approximation, i.e.

$$b_k^{(N+1)} \ge b_k^{(N)}$$

for every $N \geq 1$ and every $k \in \mathbb{Z}_{\geq 0}$.

To prove the second part, we have to modify the description of the cells. It is also possible to describe the length of the segments in the coefficient quiver relatively to the shortest segment. This parametrisation is independent of the approximation wherein we consider the cell.

For a fixed dimension k, the biggest difference between the shortest and longest segment in a cell is achieved if all k holes in the coefficient quiver are below the starting point of one long segment and above the starting points of all other segments. If this k-dimensional cell is contained in an approximation to the parameter N, all other possible k-dimensional cells are contained in this approximation. Hence $b_k^{(N)}$ is already maximal for this approximation and Part (ii) of the proposition follows.

It remains to determine the minimal number N_k such that the cell of this type is contained in the approximation to the parameter N_k. Equivalently we can compute the maximal k such that the special k-dimensional cell as introduced above is contained in the finite approximation $\mathrm{Gr}_N^a(\widehat{\mathfrak{gl}}_n)$.

We set the longest segment to N and have to distribute to remaining length of $N(n-1)$ to the remaining $2n-1$ segments such that all starting points are as low as possible. If

$$\frac{N(n-1)}{2n-1}$$

is an integer we set the length of all remaining segments to this number. For the computation of an upper bound to k we can set the starting point of the long segment as high as possible. The number of holes in the coefficient quiver which are below this starting point computes as

$$(2n-1)\Big(N - \frac{N(n-1)}{2n-1}\Big) = N(2n-1) - N(n-1) = Nn.$$

By construction there are no holes below the ohter starting points such that this is also the dimension of the corresponding cell.

In the setting that the fraction above is no integer we define

$$\ell := \Big\lfloor \frac{N(n-1)}{2n-1} \Big\rfloor \text{ and } q := N(n-1) - (2n-1)\ell \leq 2n-1.$$

We set the length of the q lower segments to $\ell+1$ and the $2n-1-q$ segments above them get the length ℓ. The highest segment again has length N. This describes a cell in the approximation to the parameter N because

$$\begin{aligned}(2n-1-q)\ell + q(\ell+1) &= (2n-1)\ell - q\ell + q\ell + q = (2n-1)\ell + q \\ &= (2n-1)\ell + N(n-1) - (2n-1)\ell = N(n-1).\end{aligned}$$

By counting the holes below the starting points of the segments, the dimension of the corresponding cell C computes as

$$\begin{aligned}\dim C &= (2n-1)\big(N-(\ell+1)\big) + 2n - 1 - q \\ &= (2n-1)\big(N-(\ell+1)\big) + 2n - 1 - \big(N(n-1) - (2n-1)\ell\big) \\ &= (2n-1)N - (2n-1)\ell + (2n-1)\ell - (2n-1) + (2n-1) - N(n-1) \\ &= (2n-1)N - N(n-1) = Nn.\end{aligned}$$

Here the number in the first row counts the holes below the starting points of the long segment. There are no holes below the other starting points by construction of this cell.

Accordingly all k-dimensional cells are contained in the approximation to the parameter N if and only if $k \leq nN$. This proves that the number of k-dimensional cells in the approximation to the parameter N is constant for

$$N \geq N_k := \lceil k/n \rceil.$$

□

This enables us to compute the number of k-dimensional cells in the degenerate affine Grassmannian by counting the k-dimensional cells of its finite approximation to the parameter N_k. For this approach it is essential that the embeddings of the quiver Grassmannians which provide the maps for the ind-variety structure of the degenerate affine Grassmannian preserve the structure and the dimension of the cells in the approximations. Accordingly there is a one to one correspondence between k-dimensional cells of the degenerate affine Grassmannian and k-dimensional cells in the approximation to the parameter $N \geq N_k$.

Proof of Theorem 5.36. Every Poincaré polynomial $p_{n,N}(q)$ of a finite approximation can be written as

$$p_{n,N}(q) = \sum_{k=0}^{Nn^2} b_k^{(N)} q^k$$

where $b_k^{(N)}$ is the number of k-dimensional cells in the approximation $\mathrm{Gr}_N^a(\widehat{\mathfrak{gl}}_n)$.

By Proposition 5.37 we know that there exists an integer N_k such that the number $b_k^{(N)}$ does not change for any $N \geq N_k$. We define $b_k := b_k^{(N_k)}$ and obtain that the Poincaré series of the affine Grassmannian is given by

$$p_n(q) = \sum_{k \geq 0} b_k q^k.$$

It remains to show that every b_k is equal to the number of partitions of k into at most $2n-1$ pieces. For this purpose we construct maps between cells of dimension k and partitions of k into at most $2n-1$ pieces. Then we check that this correspondence is bijective.

As shown above the cells are parametrised by the length of the $2n$ subsegments in the coefficient quiver of M_N. Their dimension is given by the number of holes below the starting points in the coefficient quiver. For the j-th segment this number is given by the function h_j which was determined in Proposition 5.8.

We can associate a partition of k to every cell of dimension k with parts given by the number of holes below every starting point. Going from the highest to the lowest starting point the size of the parts is descending. This partition is obtain by computing the numbers of holes h_j for all segments and ordering them from big to small.

At least the part of one of the $2n$ subsegments has to be zero. If there is a segment of length zero, there is nothing to show. If a cell corresponds to a tuple of non-zero subsegments in all $2n$ segments of $A_N \otimes \mathbb{C}^{2n}$, there can not be any free point below the starting point of the shortest segment. This follows from the shape of the coefficient quiver of $A_N \otimes \mathbb{C}^{2n}$. Let i be the index of this segment and let p_j denote the length of the segment. For all $i \in [2n]$ we know that $p_i \geq p_j$ since the j-th segment was assumed to be the shortest.

Accordingly the function $\mathrm{h}_j(\mathbf{p})$ which counts the holes below the starting point of the j-segment computes as

$$\begin{aligned}\mathrm{h}_j(\mathbf{p}) = \max\left\{0, (2n-1)p_j - j + 1\right\} \\ - \Big(\sum_{i>j} \min\{p_i, p_j\} + \sum_{i<j} \min\left\{p_i, \max\{p_j - 1, 0\}\right\} \Big) \\ = (2n-1)p_j - j + 1 - \Big(\sum_{i>j} p_j + \sum_{i<j} (p_j - 1) \Big) = 0\end{aligned}$$

because

$$\begin{aligned}\sum_{i>j} p_j + \sum_{i<j}(p_j - 1) &= (2n-j)p_j + (j-1)(p_j-1) \\ &= (2n-1)p_j - (j-1)p_j + (j-1)(p_j-1) \\ &= (2n-1)p_j - j + 1.\end{aligned}$$

Hence we can associate a partition of k into at most $2n-1$ parts to every cell. Moreover this computation shows that $\mathrm{h}_j = 0$ for all shortest segments and not only for the one with the lowest starting point.

Given two distinct cells in the same approximation the hole sequences have to vary at some point because the number of holes below the starting points determine the relative position of the starting points. If we assume that the segments correspond to cells in the same approximation the overall length of the segments is fixed and the relative positions of the starting points are sufficient to determine the whole cell. Thus we have an injective map from cells to partitions. For a given partition we now want to reconstruct its cell. This is based on the relative positions of the starting points as determined by the hole sequence.

Let $p = (p_1, \ldots, p_s)$ with $p_1 \geq p_2 \geq \cdots \geq p_s$ be a partition of k in to $s \leq 2n-1$ pieces. From this data we construct a cell descending from the highest point in the coefficient quiver. Below this point we leave $p_1 - p_2$ holes before the starting point of the next longest segment is inserted in the picture.

If $p_1 - p_2 \geq 2n-1$ there is a marked point in between corresponding to the segment we just started coming trough the sequence of holes. Below the second marked point we leave $p_2 - p_3$ holes and now have to keep record of the two segments we started which could possibly interrupt the sequence of holes.

In the same way we continue up to the starting point of the segment with p_s holes below it. After the last holes we mark the number of points missing to get a cell in the Grassmannian with Nn marked points. Every step in this process is well defined and there can not be two partitions which lead to the same cell under this process.

This establishes the bijection since we have two injective maps between cells and partitions which are inverse to each other. The generating function for partitions of k into at most $2n-1$ pieces is known to be

$$\prod_{k=1}^{2n-1} (1-q^k)^{-1}.$$

□

REMARK. The parametrisation of the cells by certain partitions should allow to describe the Poincaré polynomials of the finite approximations explicitly. Unfortunately we have not found any polynomial condition to decide whether a partition belongs to the approximation for the parameter N or not. So far we only have a condition which is based on the algorithm in the proof above. This method is discussed in the next section where we also apply the algorithm in some examples.

THEOREM 5.38. *The Euler Poincaré series $p_n^k(q)$ of the partial degenerate affine Grassmannian*

$$\mathrm{Gr}^k\big(\,\widehat{\mathfrak{gl}}_n\,\big)$$

is given by

$$p_n^k(q) := \prod_{r=1}^{n+k-1} (1-q^r)^{-1}.$$

PROOF. Similar as above we obtain a stabilising condition of the coefficients in the Poincaré polynomials of the approximations because the map Φ_N^k preserves the dimension of cells whose segments are not longer than N. We can use the functions h_j with $m = n+k$ to compute the dimension of these cells. For every dimension r there exists an N such that all r-dimensional cells in the approximation to the parameter N are expressible using segments of length at most N.

In this setting the computation of b_r and the inverse map from partitions back to cells is analogous to the special case as described above. All steps in the proof work similar and we can identify the cells in the partial degenerations with partitions of the cell dimension into $n+k-1$ parts.

The singular homology commutes with direct limits [**53**, p. 399]. The Betti numbers of the finite dimensional approximations are computed following [**33**, §B.3 Lemma 6] and [**70**, Chapter 5, §5]. □

For the affine Grassmannian we recover the same formula for the Poincaré series as with the classical computation based on the length of the elements in the affine Weyl group [**7**].

REMARK. The quiver Grassmannian

$$\mathrm{Gr}_{2Nn}^{A_{2N}}\big(A_{2N} \otimes \mathbb{C}^{2n}\big)$$

provides the finite approximations

$$\mathrm{Gr}_N\big(\,\widehat{\mathfrak{gl}}_{2n}\,\big) \text{ and } \mathrm{Gr}_{2N}^a\big(\,\widehat{\mathfrak{gl}}_n\,\big)$$

such that both approximations have the same Poincaré polynomial. But the identifications of the quiver Grassmannian with the approximations are different such that one limit is the affine Grassmannian and the other is the Feigin-degenerate affine Grassmannian.

5.9. Partitions and Cells in the Quiver Grassmannians

In this section we develop a method to decide whether a partition corresponds to a cell in the approximation to the parameter N or not. Let $\mathrm{P}_{2n-1}(k)$ be the set of all partitions of k into at most $2n-1$ pieces. The map ψ is defined as

$$\psi : \mathcal{C}_{nN}^{(2n)}(\Delta_1, \mathrm{I}_N) \to \bigcup_{k=0}^{Nn^2} \mathrm{P}_{2n-1}(k)$$
$$\mathbf{p} \longmapsto \lambda(\mathbf{p})$$

where $\lambda(\mathbf{p})$ is the partition which is obtained by ordering the numbers $\mathrm{h}_j(\mathbf{p})$ for $j \in [2n]$. The goal of this section is to describe the image

$$\mathrm{P}_{2n-1}^{N}(k) := \psi\Big(\mathcal{C}_{nN}^{(2n)}\Big) \cap \mathrm{P}_{2n-1}(k)$$

explicitly. With these sets the Poincaré polynomial of the approximation computes as

$$\mathrm{p}_{n,N}(q) = \sum_{k=0}^{Nn^2} \big|\mathrm{P}_{2n-1}^{N}(k)\big| q^k.$$

The maps between cells and partitions generalise to the class of quiver Grassmannians we introduced in the beginning of this chapter. Hence we can compute the Poincaré polynomial of them as well.

Starting with a partition $\lambda \in \mathrm{P}_{x+y-1}(k)$ we start to compute the corresponding cell following the algorithm described in the proof of Theorem 5.36. After we have drawn the first black dot below the last hole, we count the number of black dots above the lowest hole and denote this number by λ_1^*.

This is the only way to compute this number that we have found so far. It is desirable to find a direct computation for this number which is independent of the algorithm to compute cells from partitions.

THEOREM 5.39. *The Poincaré polynomial* $\mathrm{p}_{x,y,N}(q)$ *of the quiver Grassmannian*

$$\mathrm{Gr}_{xN}^{A_N}\big(A_N \otimes \mathbb{C}^{x+y}\big)$$

is given by

$$\mathrm{p}_{x,y,N}(q) = \sum_{k=0}^{Nxy} \big|\mathrm{P}_{x,y}^{N}(k)\big| q^k.$$

where

$$\mathrm{P}_{x,y}^{N}(k) := \Big\{\lambda \in \mathrm{P}_{x+y-1}(k) : \lambda_1 \le yN, \lambda_1^* \le xN\Big\}.$$

PROOF. For a cell $\mathbf{p}$ in $\mathcal{C}_{xN}^{(x+y)}$ the complement $\mathbf{p}^*$ is defined by setting

$$p_j^* := N - p_{x+y+1-j}$$

for all $j \in [x+y]$. Hence the complement $\mathbf{p}^*$ lies in $\mathcal{C}_{yN}^{(x+y)}$. If we apply the map ψ to the cell $\mathbf{p}^*$, the first part in the partition $\lambda(\mathbf{p}^*)$ is given by the λ_1^* as constructed above. The map ψ in this setting is defined analogous to the special case $x = y$.

A cell $\mathbf{p}$ in $\mathcal{C}_{xN}^{(x+y)}$ consists of xN marked points and yN unmarked points in the coefficient quiver of $A_N \otimes \mathbb{C}^{x+y}$. Thus under each starting point of a segment there can be at most yN unmarked points such that we obtain $\lambda_1 \le yN$. For the complementary cell $\mathbf{p}^*$ in $\mathcal{C}_{yN}^{(x+y)}$ we obtain $\lambda_1^* \le xN$ in the same manner. Accordingly we have a necessary condition to describe the image, i.e.

$$\psi\Big(\mathcal{C}_{xN}^{(x+y)}\Big) \cap \mathrm{P}_{x+y-1}(k) \subseteq \mathrm{P}_{x,y}^{N}(k).$$

Given the information that for a partition λ it is enough to take at most xN many marked points to cover the k many holes below the highest starting point, we can add the remaining marked points below the lowest hole and the remaining unmarked points above the highest starting point. What we obtain is a cell with exactly xN marked points and yN unmarked points. This cell is contained in the set $\mathcal{C}_{xN}^{(x+y)}$. Hence the condition we imposed is sufficient, i.e.

$$\psi\left(\mathcal{C}_{xN}^{(x+y)}\right) \cap \mathrm{P}_{x+y-1}(k) = \mathrm{P}_{x,y}^{N}(k).$$

□

EXAMPLE 5.40. In the set $\mathrm{P}_3(5)$ we have the partition $\lambda := (3, 1, 1)$. Now we describe the steps in the algorithm to compute the corresponding cell. Here we draw the intermediate steps horizontal in order to reduce the space we need for the pictures. We start with one marked box and $3 - 1$ holes, i.e.:

● ○ ○

After $4 = 2n$ dots we draw a horizontal line to indicate that we are at the next length level of the coefficient quiver and that we have to take track of the points which have been marked already. These have to be repeated now with a period of $2n$. The next box is marked again. Then we draw the first separator and keep track of the first repetition. Then we draw the last starting point:

● ○ ○ ● $\longmapsto$ ● ○ ○ ●| $\longmapsto$ ● ○ ○ ●|●

Then we draw the last starting point and add the final hole:

● ○ ○ ●|● ● $\longmapsto$ ● ○ ○ ●|● ● ○

The next dot would have to be black again but we do not have to add it. The current diagram has 4 marked and 3 unmarked points. By Theorem 5.39 we know that it can be turned into a cell for $N = 2$. We move the separator one to the left and add a white dot on the left.

● ○ ○|● ● ● ○| $\longmapsto$ |○ ● ○ ○|● ● ● ○|

The resulting diagram corresponds to a 5-dimensional cell in the quiver Grassmannian

$$\mathrm{Gr}_4^{A_2}\left(A_2 \otimes \mathbb{C}^4\right)$$

since it has 4 marked and 4 unmarked points and $5 = 3 + 1 + 1$ holes below the starting points of the segments.

The partition $\lambda = (3, 2, 1)$ is contained in the set $P_3(6)$. The steps in the algorithm to compute the corresponding cell are as follows. We start with one black dot followed by $3 - 2$ white dots. Then we have a black dot again which is followed by $2 - 1$ white dots:

● $\longmapsto$ ● ○ $\longmapsto$ ● ○ ● $\longmapsto$ ● ○ ● ○

We arrived at 4 dots so we have to draw the first separator, keep track of repetitions and add the last starting point:

● ○ ● ○| $\longmapsto$ ● ○ ● ○|● $\longmapsto$ ● ○ ● ○|● ●

Then there is the next repetition before we can add the last hole:

● ○ ● ○|● ● ● $\longmapsto$ ● ○ ● ○|● ● ● ○

In this diagram we have 5 marked and 3 unmarked points. By Theorem 5.39 we know that it is not possible to turn this diagram into a cell for the parameter $N = 2$. To make it a cell for the parameter $N = 3$ we have to add one black dot on the right and three white dots on the left. Finally we move the separators in the right positions:

$$\bullet \quad \circ \quad \bullet \quad \circ \,|\, \bullet \quad \bullet \quad \bullet \quad \circ \,|\, \bullet$$

$$\downarrow$$

$$\circ \quad \circ \quad \circ \,|\, \bullet \quad \circ \quad \bullet \quad \circ \,|\, \bullet \quad \bullet \quad \bullet \quad \circ \,|\, \bullet$$

$$\downarrow$$

$$|\, \circ \quad \circ \quad \circ \quad \bullet \,|\, \circ \quad \bullet \quad \circ \quad \bullet \,|\, \bullet \quad \bullet \quad \circ \quad \bullet \,|$$

This diagram corresponds to a 6-dimensional cell in the quiver Grassmannian

$$\mathrm{Gr}_6^{A_3}\left(A_3 \otimes \mathbb{C}^4\right)$$

because there are 6 marked and 6 unmarked points and $6 = 3 + 2 + 1$ holes below the starting points of the segments.

CHAPTER 6

The Degenerate Affine Flag Variety

In this chapter we define affine flag varieties and describe the link between affine flag varieties and quiver Grassmannians. For more details about the general construction of affine flags see Chapter XIII in the book by S. Kumar [**53**].

Definition 6.1. Let $\widehat{G}$ be the Kac-Moody group corresponding to the affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$. For a parahoric subgroup P of $\widehat{G}$ with $I \subset P$ where I is the standard Iwahori subgroup of $\widehat{G}$ the **affine flag variety** is defined as

$$\mathcal{F}l_P\big(\widehat{\mathfrak{g}}\big) := \widehat{G}/P.$$

If the parahoric subgroup is equal to the standard Iwahori subgroup, we simply write $\mathcal{F}l\big(\widehat{\mathfrak{g}}\big)$ and refer to this flag variety as (standard) flag variety. For the affine Lie algebras $\widehat{\mathfrak{gl}}_n$ and $\widehat{\mathfrak{sl}}_n$ it is possbile to identify their flag varieties with the set of full periodic lattice chains and the set of special lattice chains respective. This is shown for example in the survey by U. Görtz [**37**] and the articles by A. Beauville and Y. Laszlo [**3**] and U. Görtz [**36**]. V. Kac and D. Peterson described representations of these groups using Sato Grassmannians [**45**]. This leads to the subsequent parametrisation of the affine flag variety and is related to the lattice chain description.

For $\ell \in \mathbb{Z}$ let V_ℓ be the vectorspace

$$V_\ell := \operatorname{span}(v_\ell, v_{\ell-1}, v_{\ell-2}, \dots)$$

which is a subspace of the infinite dimensional $\mathbb{C}$-vectorspace V with basis vectors v_i for $i \in \mathbb{Z}$. The Sato Grassmannian for $m \in \mathbb{Z}$ is defined as

$$\mathrm{SGr}_m := \big\{ U \subset V \ : \ \text{There exists a } \ell < m \text{ s.t. } V_\ell \subset U \text{ and } \dim U/V_\ell = m - \ell \ \big\}.$$

Remark. For an n-dimensional vector space W over $\mathbb{C}$ with basis $w_1, \dots, w_n$ we can identify the space $W \otimes \mathbb{C}[t, t^{-1}]$ with V by

$$v_{n(k-1)+j} = w_j \otimes t^{-k}.$$

This gives an embedding $\widehat{\mathfrak{gl}}_n \subset \mathfrak{gl}_\infty$ and allows to describe the affine flag variety of type $\mathfrak{gl}_n$ inside the full infinite flag variety of type A_∞. For more details on this construction see [**30**] and [**45**].

Here we give an equivalent description for the affine flag variety which is independent of this identification of the basis. It is shown in Section 5.2 that both parametrisations are equivalent for the affine Grassmannian. The equivalence for the affine flag variety is shown in the same way.

PROPOSITION 6.2. The **affine flag variety** $\mathcal{F}l\big(\widehat{\mathfrak{gl}}_n\big)$ as subset in the product of Sato Grassmannians is parametrised as

$$\mathcal{F}l\big(\widehat{\mathfrak{gl}}_n\big) \cong \left\{ \big(U_k\big)_{k=0}^{n-1} \in \prod_{k=0}^{n-1} \mathrm{SGr}_k \ : \ U_0 \subset U_1 \subset \ldots \subset U_{n-1} \subset s_n U_0 \right\}.$$

It is shown by E. Feigin in [**29**] that the degeneration of the classical flag variety he introduced in [**28**] admits a description via vector space chains where the spaces are related by projections instead of inclusions. This construction is used to identify the classical flag variety and its degenerations with quiver Grassmannians for an equioriented quiver of type A by G. Cerulli Irelli, E. Feigin and M. Reineke in [**20**]. The observation that the Feigin degeneration of the flag variety admits a description where the inclusion relations of the vectorspaces are relaxed gives rise to more general degenerations where arbitrary linear maps are allowed between the vector spaces. These degenerations of the flag variety are called linear and are studied in [**19**]. Here we want to follow the same approach and degenerate the affine flag variety by replacing the inclusion relations for the chains of vector spaces with projections. In later sections of this chapter we discuss a more general approach to degenerate this flag variety. We show that some of the methods to study the degenerate flag variety of this section still apply in the more general setting.

DEFINITION 6.3. The **degenerate affine flag variety** $\mathcal{F}l^a\big(\widehat{\mathfrak{gl}}_n\big)$ is defined as

$$\mathcal{F}l^a\big(\widehat{\mathfrak{gl}}_n\big) := \left\{ \big(U_k\big)_{k=0}^{n-1} \in \prod_{k=0}^{n-1} \mathrm{SGr}_k \ : \ pr_{k+1} U_k \subset U_{k+1}, \ pr_n U_{n-1} \subset s_n U_0 \right\}.$$

Here pr_i is the projection of v_i to zero which corresponds to the projection of $w_i \otimes 1$ to zero by the identification of the basis we make above. This degeneration can also be called the Feigin-degenerate affine flag variety since its definition is motivated by the definition for the linearly oriented type A quiver which can be used to define quiver Grassmannians which are isomorphic to the Feigin degeneration of the classical flag varieties [**28, 29**].

6.1. Finite Approximation by Quiver Grassmannians

For a positive integer ω we define the **finite approximation** of the degenerate affine flag variety as

$$\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big) := \Big\{ \big(U_i\big)_{i=0}^{n} \in \mathcal{F}l^a\big(\widehat{\mathfrak{gl}}_n\big) \ : \ V_{-\omega n} \subseteq U_0 \subseteq V_{\omega n} \Big\}.$$

Analogous we define $\mathcal{F}l_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ for the non-degenerate affine flag variety.

THEOREM 6.4. *Let $\omega \in \mathbb{N}$ be given, take the nilpotent quiver representations $X_\omega := Y_\omega := \bigoplus_{i\in\mathbb{Z}_n} U_i(\omega n)$ and let $\mathbf{e}_\omega$ denote the dimension vector of X_ω, i.e. $\mathbf{e}_\omega := \mathbf{dim}\, X_\omega$. Then*

$$\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big) \cong \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega).$$

REMARK. This identification allows us to use all the theory and the results developed for quiver Grassmannians in order to study the Feigin-degenerate affine flag variety. The rest of this Chapter will deal with the proof of this theorem and the consequences from applying the theory developed for quiver Grassmannians and varieties of quiver representations in Chapter 3 and Chapter 4.

For the proof of the theorem we need a different labelling of the basis elements for the vector spaces over the vertices of the representation $X_\omega \oplus Y_\omega$ than the one we introduce in Section 4.4. This leads to a different arrangement of the segments in the coefficient quiver which captures the structure of the maps between the elements of the Sato Grassmannians.

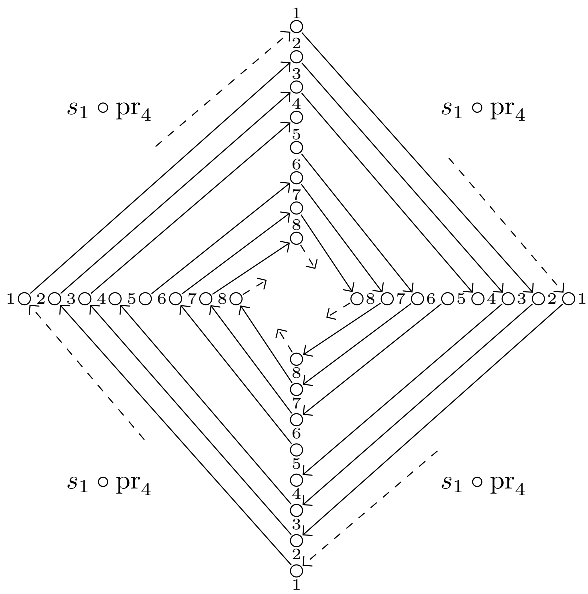
PROPOSITION 6.5. The quiver representation $X_\omega \oplus Y_\omega$ is isomorphic to the quiver representation

$$M_\omega := \big(M_{\alpha_i} := s_1 \circ \mathrm{pr}_{\omega n}\big)_{i \in \mathbb{Z}_n}.$$

PROOF. The vertices of Δ_n are in bijection with the set $\mathbb{Z}_n$ and we choose the representatives $0, 1, \ldots, n-1$. For the representation $X_\omega \oplus Y_\omega$ the vectorspace over each vertex $i \in \mathbb{Z}_n$ has dimension $2\omega n$. For the arrangement of the segments in the coefficient quiver corresponding to the summands $U_i(\omega n)$ of $X_\omega \oplus Y_\omega$ as in Section 4.4 we obtain the maps $s_2 : \mathbb{C}^{2\omega n} \to \mathbb{C}^{2\omega n}$ along the arrows of Δ_n since there are starting two segments of length ωn over each vertex.

Now we rearrange the segments. Over each vertex let one segment start in the first basis vector and map to the second basis vector over the next vertex. For a segment starting over the vertex $i \in \mathbb{Z}_n$ in the k-th step of the segment the arrow in the coefficient quiver goes from the k-th basis element over the vertex $i + k - 1$ to the $k + 1$-th basis element over the vertex $i + k$. The segment ends in the ωn-th basis vector over the vertex $i + n = i - 1$. The second segment over each vertex starts in the basis vector $\omega n + 1$. For the vertex $i \in \mathbb{Z}_n$ in the k-th step of this segment the arrow in the coefficient quiver goes from the $\omega n + k$-th basis element over the vertex $i+k-1$ to the $\omega n+k+1$-th basis element over the vertex $i+k$. The segment ends in the $2\omega n$-th basis vector over the vertex $i + n = i - 1$. The maps between the copies of $\mathbb{C}^{2\omega n}$ over the vertices corresponding to this arrangement of the segments are given by $s_1 \circ \mathrm{pr}_{\omega n}$. □

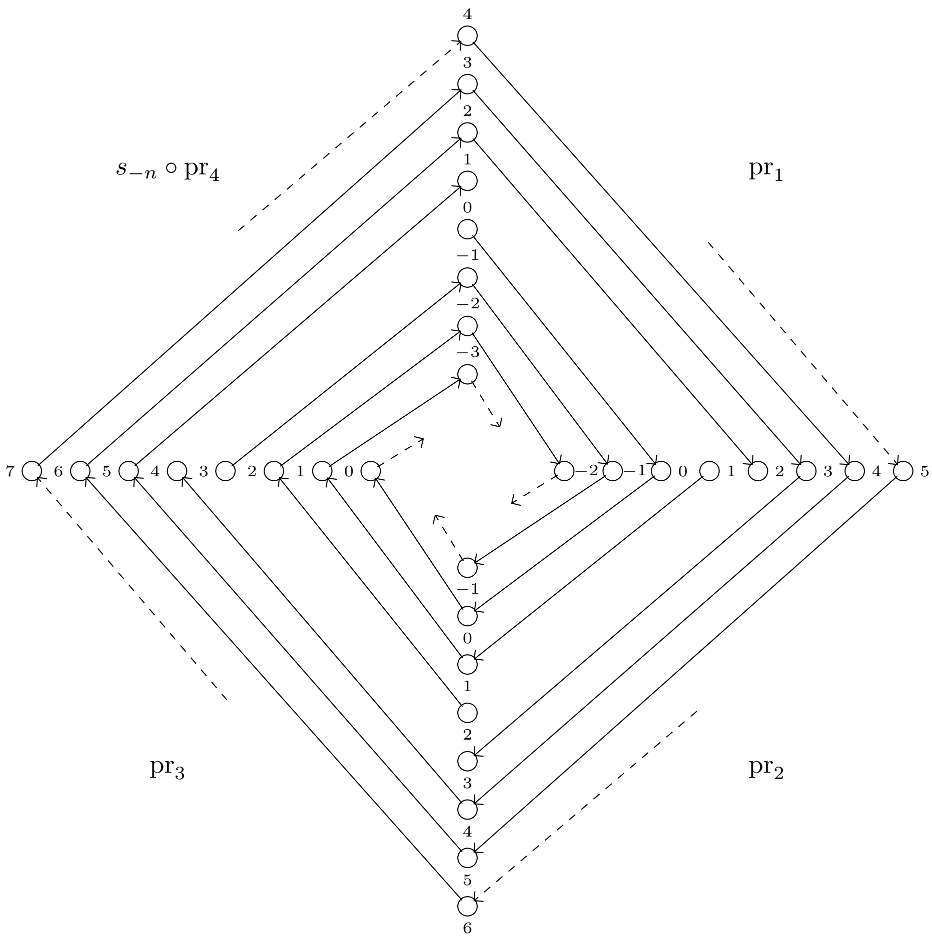
EXAMPLE 6.6. For $n = 4$ and $\omega = 1$ the coefficient quiver of $M_\omega \cong X_\omega \oplus Y_\omega$ is of the form



The dashed arrows in the picture indicate where the segments grow if we increase the value of ω.

Now we want to change the indices of the basis vectors over the vertices $k \in \Delta_n$ in order to match the indices of the basis vectors for the spaces in the Sato Grassmannians SGr_k. We keep the structure of the arrows between the vertices in the coefficient quiver as introduced above and just change the labelling of the points in the coefficient quiver. For the vertex $k \in \Delta_n$ we label the first point in the coefficient quiver over k by $\omega n+k$ the second point gets the label $\omega n+k-1$ and the last one will have the label $-\omega n+k+1$ since there are $2\omega n$ points over each vertex of Δ_n. With this labelling the maps along the arrows of Δ_n for $k \in \{0,1,\dots,n-1\}$ are given by pr_{k+1}. For $k=n$ we obtain the map $s_{-n} \circ \mathrm{pr}_n$. This is computed from the index shift of the points over each vertex in the coefficient quiver.

Example 6.7. Before we turn attention to the proof of the theorem we consider the coefficient quiver of $M_\omega \cong X_\omega \oplus Y_\omega$ for $n=4$ and $\omega=1$ where we used the new labelling of the points in the coefficient quiver.



Again the dashed arrows in the picture indicate where the segments grow if we increase the value of ω.

Proof of Theorem 6.4. In the coefficient quiver of $M_\omega := X_\omega \oplus Y_\omega$ we arrange the segments corresponding to projective and injective summands as in the example above. Hence the maps between the vector spaces over the vertices coincide with the maps between the spaces in the degenerate affine flag variety.

To finish the proof we have to identify the spaces in the finite approximations of the Sato Grassmannians which correspond to points in the degenerate affine flag variety with the vector spaces corresponding to the elements of the quiver Grassmannian. In Proposition 5.13 we have identified the approximation $\mathrm{SGr}_{m,\ell}$ of the Sato Grassmannian SGr_m with the classical Grassmannian $\mathrm{Gr}_{m+\ell}(2\ell)$. For the Sato Grassmannian SGr_0 and $\ell = \omega n$ we obtain the isomorphism $\mathrm{SGr}_{0,\omega n} \cong \mathrm{Gr}_{\omega n}(2\omega n)$. This identifies the vector space over the first vertex of the quiver Δ_n for a representation in the quiver Grassmannian with the first space in the tuple of vector spaces parametrising a point in the approximation of the degenerate affine flag variety.

The cyclic relations of the vector spaces describing a point $(U_i)_{i\in\mathbb{Z}_n}$ in the degenerate affine flag variety induce the following restrictions for the approximation to the parameter ω

$$V_{-n\omega+i} \subset U_i \subset V_{n\omega+i} \quad \text{for } U_i \in \mathrm{SGr}_i.$$

Accordingly the corresponding approximations of the Sato Grassmannians SGr_i are also isomorphic to the classical Grassmannian $\mathrm{Gr}_{\omega n}(2\omega n)$. The points in the approximation

$$\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big)$$

are described by tuples consisting of vector spaces $U_i \in \mathrm{SGr}_i$ which are subject to the above bounding condition and are compatible with the maps pr_{i+1} and $s_{-n} \circ \mathrm{pr}_n$. Hence they are in bijection with points in the quiver Grassmannian

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega)$$

because the points in the quiver Grassmannian are described as tuples of vectorspaces $V_i \in \mathrm{Gr}_{\omega n}(2\omega n)$ for $i \in \mathbb{Z}_n$ which are compatible with the maps $s_1 \circ \mathrm{pr}_{\omega n}$. The isomorphism of the approximations of the Sato Grassmannians and the classical Grassmannian induce the correspondence of these maps with the maps between the Sato Grassmannians as used for the definition of the degenerate affine flag variety. Here the explicit coordinate description of this isomorphism is obtained as in the examples above. □

In the rest of the chapter we apply the results about quiver Grassmannians to the approximations of the affine flag variety and its degenerations.

Corollary 6.8. $\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ is a projective variety of dimension $\omega \cdot n^2$.

Proof. Let $k := \sum_{i\in\mathbb{Z}_n} x_i$ where x_i is the multiplicity of $U_i(\omega n)$ as summand of X and $m := k + \sum_{i\in\mathbb{Z}_n} y_i$ where y_i is the multiplicity of $U_{i-\omega n+1}(\omega n)$ as summand of Y. From Lemma 3.22 we obtain

$$\dim \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega) = \omega k(m-k).$$

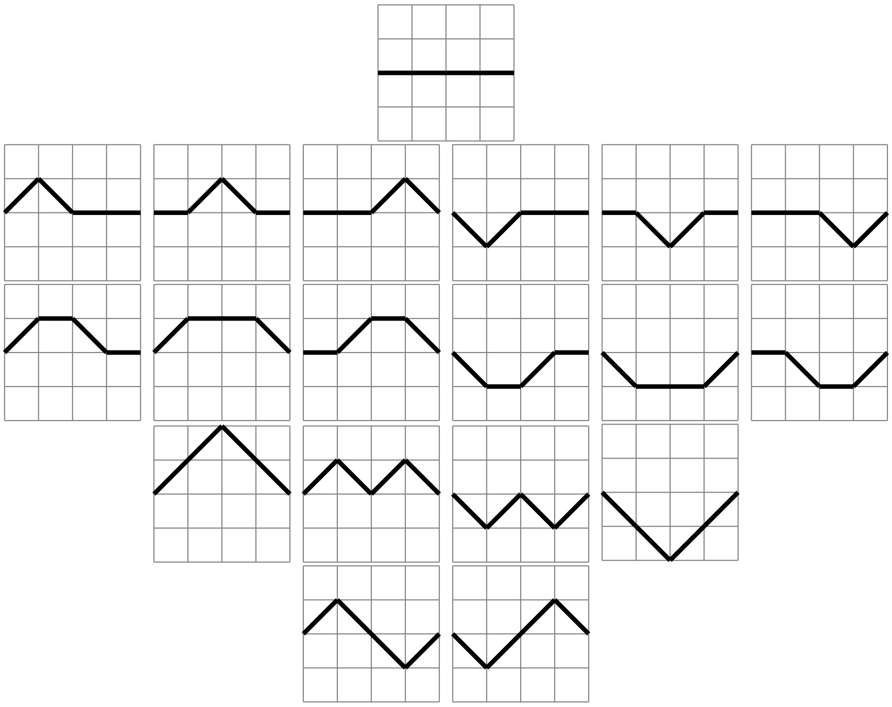
In this special case we have $k = m - k = n$ and thus $\dim \mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big) = \omega \cdot n^2$. It is a projective variety because all quiver Grassmannians for nilpotent representations of Δ_n are projective varieties. □

6.2. Irreducible Components and Grand Motzkin Paths

In this section we use the formula for the irreducible components of the quiver Grassmannians as developed in Lemma 3.23 to describe the irreducible components of the approximations of the degenerate affine flag variety.

A **grand Motzkin path** of length n is a path on the grid $\mathbb{Z}^2$ from $(0,0)$ to $(n,0)$ where the allowed steps are $(1,0),(1,1),(1,-1)$. Is possible to reconstruct a path from the n-tuple consisting of the second entries of the steps it takes.

EXAMPLE 6.9. For $n = 4$ there are 19 grand Motzkin paths



which are in bijection with the tuples

$$(0,0,0,0),$$
$$(1,-1,0,0),\ (0,1,-1,0),\ (0,0,1,-1),\ (-1,1,0,0),\ (0,-1,1,0),\ (0,0,-1,1),$$
$$(1,0,-1,0),\ (1,0,0,-1),\ (0,1,0,-1),\ (-1,0,1,0),\ (-1,0,0,1),\ (0,-1,0,1),$$
$$(1,1,-1,-1),\ (1,-1,1,-1),\ (-1,1,-1,1),\ (-1,-1,1,1),$$
$$(1,-1,-1,1),\ (-1,1,1,-1).$$

The sum over the entries in the tuples has to equal zero to satisfy that the path comes back to the first axis.

Normal Motzkin paths are not allowed to go below the line between $(0,0)$ and $(n,0)$ (first axis). In this example they can be found on the left side of the list above except from the last row. The paths in the last row can not be derived from the normal Motzkin paths by reflecting the path at the first axis. The first case where these mixed paths appear is $n = 4$.

LEMMA 6.10. *The irreducible components of $\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ are in bijection with the set of grand Motzkin paths of length n.*

PROOF. By Lemma 3.23 we know that the irreducible components of the quiver Grassmannian in this setting are parametrised by

$$\Big\{\mathbf{p} \in \{0,1,2\}^n : \sum_{i\in\mathbb{Z}_n} p_i = n\Big\}.$$

We obtain a bijection with grand Motzkin paths by sending a tuple $\mathbf{p}$ describing an irreducible component to the tuple $(p_i - 1)_{i\in\mathbb{Z}_n}$.

This tuple describes a grand Motzkin path since the sum over all entries is equal to zero and the entries take values $-1, 0$ and 1. Starting with the tuple parametrising a grand Motzkin path we add one to every entry and obtain a tuple corresponding to an irreducible component because all entries are $0, 1$ or 2 and the sum of the entries is equal to n. ☐

Thus the number of irreducible components is independent of ω whereas the number of strata is growing with ω.

EXAMPLE 6.11. The grand Motzkin paths from the previous example correspond to the following list

$$\begin{gathered}(1,1,1,1),\\ (2,0,1,1),\ (1,2,0,1),\ (1,1,2,0),\ (0,2,1,1),\ (1,0,2,1),\ (1,1,0,2),\\ (2,1,1,0),\ (2,1,0,1),\ (1,2,1,0),\ (0,1,2,1),\ (1,0,1,2),\ (0,1,1,2),\\ (2,2,0,0),\ (2,0,2,0),\ (0,2,0,2),\ (0,0,2,2),\\ (2,0,0,2),\ (0,2,2,0).\end{gathered}$$

with tuples of multiplicities for the indecomposable representations $U_i(\omega n)$ describing the irreducible components. Here $(1,1,1,1)$ corresponds to the component of X_ω.

6.3. Cellular Decomposition

The finite dimensional approximations of the Feigin-degenerate affine flag variety admit a cellular decomposition as described in Theorem 4.10 in the chapter about the torus action on the quiver Grassmannians. In this section we examine how this decomposition changes by increasing the value of ω. The representation $M_\omega := X_\omega \oplus Y_\omega$ contains exactly two copies of every indecomposable representation $U_i(\omega n)$ and thus there are 2^n possibilities to embed X_ω into M_ω. Hence the stratum of X_ω decomposes into 2^n cells.

This is also the highest number of cells any stratum could have since it is only possible to have two distinct sub-segments embedded into the two segments of the coefficient quiver corresponding to the two copies of $U_i(\omega n)$. If in a stratum the segments are the same for some i, the number of cells in this stratum is strictly smaller. Given any stratum in the finite approximation it is possible to determine all of its cells and their dimension.

PROPOSITION 6.12. The base in the stratification of the finite approximation for $\omega \in \mathbb{N}$ is given by

$$B_{\mathbf{e}_\omega} = \bigoplus_{i\in\mathbb{Z}_n} U\big(i; \lfloor \omega\cdot n/2\rfloor\big) \oplus U\big(i; \lceil \omega\cdot n/2\rceil\big).$$

PROOF. Here we need the labelling of the basis elements for the vector spaces over the vertices of the quiver Δ_n corresponding to the representation $X_\omega \oplus Y_\omega$ as introduced in Chapter 4.

For every choice of ω, there are two segments in the coefficient quiver of $X_\omega \oplus Y\omega$ ending at each vertex $i \in \mathbb{Z}_n$. The i-th entry of the dimension vector $\mathbf{e}_\omega := \mathbf{dim}\, X_\omega$ is given by ωn. Thus taking the e_i inner points over the vertex $i \in \mathbb{Z}_n$ corresponds to taking $\lfloor \omega/2 \rfloor$ points in every segment of the coefficient quiver and one additional point in the inner segment corresponding to some summand $U_j(\omega n)$ if ω is not even. These j's are obtained from the i's by some shift of indices.

Since $\mathbf{e}$ is homogeneous and we do it for every vertex $i \in \mathbb{Z}_n$ the exact value of the shift does not matter. We only obtain that one of the segments of $B_{\mathbf{e}_\omega}$ ending at vertex i is longer by one if ω is odd and they are the same if ω is even.

The length of the short segments corresponding to a summand of $B_{\mathbf{e}_\omega}$ is given by $\lfloor \omega n/2 \rfloor$. For the injective labelling of the indecomposable representations this corresponds to a summand

$$U\big(i; \lfloor \omega \cdot n/2 \rfloor\big) = U_{i - \lfloor \omega \cdot n/2 \rfloor + 1}\big(\lfloor \omega \cdot n/2 \rfloor\big)$$

and we obtain them for every $i \in \mathbb{Z}_n$. Thus the long segments correspond to summands

$$U\big(i; \lceil \omega \cdot n/2 \rceil\big)$$

for every $i \in \mathbb{Z}_n$. □

Remark. For $N = \omega \cdot n$ with ω even the stratum of $B_{\mathbf{e}_\omega}$ has exactly one cell and it is zero-dimensional. For odd ω the stratum decomposes into 2^n cells and the stratum is n-dimensional. Hence we can distinguish between odd and even limits of the finite dimensional approximations.

Now we want to examine how the Euler Poincaré characteristic changes with increasing ω.

Proposition 6.13. Let χ_ω be the Euler Poincaré characteristic of $\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big)$. It is bounded as

$$\big(2\lceil \omega/2 \rceil\big)^n \quad \leq \quad \chi_\omega \quad \leq \quad \big(\omega n + 1\big)^{2n}.$$

Proof. The Euler Poincaré characteristic of $\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ is equal to the Euler Poincaré characteristic of the quiver Grassmannian $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega)$. The cells in this quiver Grassmannian are in bijection with certain successor closed subquivers in the coefficient quiver of $X_\omega \oplus Y_\omega$. These subquivers are parametrised by tuples indexed by the indecomposable direct summands in $X_\omega \oplus Y_\omega$ and the corresponding entry equals the length of the segment embedded into the segment corresponding to this summand.

Thus we obtain tuples $\mathbf{p}$ of non-negative integers in $\mathbb{Z}^{2n}$ because $X_\omega \oplus Y_\omega$ has two copies of every $U(i; \omega n)$ for $i \in \mathbb{Z}_n$ as summand. Let p_i be the length of a segment embedded into a copy of $U(i; \omega n)$. It has to satisfy $0 \leq p_i \leq \omega n$. We set $\mathbf{dim}\, U(i; 0) := 0$ and define the function

$$\begin{aligned} f_i : \{0, 1, \dots, \omega n\} &\longrightarrow \mathbb{Z}^n \\ p_i &\longmapsto \mathbf{dim}\, U(i; p_i) \end{aligned}$$

which sends p_i to the dimension vector of the indecomposable representation corresponding to the segment it describes.

Using these functions we can parametrise the cells of the quiver Grassmannian by the set

$$Z_\omega := \Big\{ \mathbf{p} \in \mathbb{Z}^{2n} \ : \ 0 \leq p_j \leq \omega n \text{ for all } j \in [2n] \text{ and } \sum_{i \in \mathbb{Z}_n} f_i(p_i) + f_i(p_{n+i}) = \mathbf{e} \Big\}.$$

Forgetting the constraint by the sum of the dimension vectors we obtain

$$|Z_\omega| \leq (\omega n + 1)^{2n}.$$

To compute a lower bound on the Euler characteristics, we give an explicit description of some cells in the Grassmannian $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega)$ and count them. For two integers p and q with $0 \leq p < q \leq \omega$ and $p + q = \omega$ define the representations

$$U(i; qn) \quad \text{and} \quad U_(i; pn)$$

which both embed into $U(i; \omega n)$ by definition.

Their direct sum $U(i; qn) \oplus U(i; pn)$ has the same dimension vector as $U(i; \omega n)$. The parameter p can be computed as $p = \omega - q$ and the pairs (p, q) satisfying the conditions above are obtained from the q's satisfying

$$\lfloor \omega/2 \rfloor + 1 \leq q \leq \omega.$$

Hence there are $\lceil \omega/2 \rceil$ many of them. In that way we can choose one parameter q_i for every $i \in Z_n$ and define the representation

$$V(\mathbf{q}) := \bigoplus_{i \in \mathbb{Z}_n} U(i; q_i n) \oplus U(i; (\omega - q_i)n)$$

which is by construction an element of the quiver Grassmannian $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega)$.

There are $\lceil \omega/2 \rceil^n$ strata corresponding to the representations $V(\mathbf{q})$ and each of them decomposes into 2^n cells since $p_i \neq q_i$ for every $i \in \mathbb{Z}_n$ and this gives us 2^n distinct possibilities to embed the segments of $V(\mathbf{q})$ into the segments in the coefficient quiver of $X_\omega \oplus Y_\omega$. □

This shows that with increasing ω the Euler characteristic grows at least with ω^n whereas the dimension of the approximation is growing only linearly. In Section 6.10 we give a formula for the Poincaré polynomials of the approximations which is based on the parametrisation of the cells by successor closed subquivers.

Remark. For $n \in [5]$ we can compute the Euler characteristic χ_1 and the number of strata in the quiver Grassmannian $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_1}(M_1)$ using the computer program from Appendix B.1. For bigger ω a normal computer can only handle the data for even smaller n. In the following table we list these numbers together with the number of irreducible components which are parametrised as in Lemma 6.10.

n	1	2	3	4	5
$\lvert C_n(\mathbf{d}) \rvert$	1	3	7	19	51
$\lvert \mathrm{Strata}_1 \rvert$	1	6	41	585	12603
χ_1	2	15	226	6137	265266
χ_2	3	65	3511	359313	...
χ_3	4	175	20620	...	...

These first values of χ_1 indicate that the Euler characteristics for $\omega = 1$ is growing faster even than $(2n)^n$ with increasing n.

6.4. Affine Dellac Configurations

For the Feigin degeneration of the classical flag variety of type A_n, the Poincaré polynomial can be computed using Dellac configurations which are counted by the median Genocchi numbers. This description was develloped by E. Feigin in [**29**]. The torus fixed points of the symplectic degenerated flag variety are identified with symplectic Dellac configurations by X. Fang and G. Fourier in [**26**]. In this section we introduce affine Dellac configurations which turn out to be in bijection with the cells of the Feigin-degenerate affine flag variety. This identification is based on the parametrisation of the cells via sucessor closed subquivers.

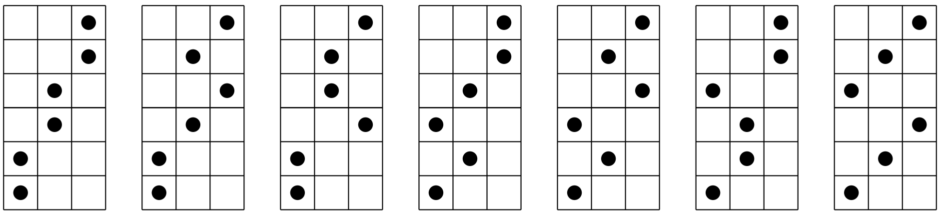
6.4.1. Classical Dellac Configurations. First we recall the definition of the classical Dellac configuration and show the idea behind the identification with the cells of the degenerate flag variety $\mathcal{F}l^a(\mathfrak{sl}_n)$. This will help us to find the right analogue to classical Dellac configurations in the affine setting.

Definition 6.14. In a rectangle of $2n \times n$ boxes a **Dellac configuration** D consists of $2n$ marked boxes such that:

(1) each row contains exactly one marked box,
(2) each column contains exactly two marked boxes,
(3) the index $(r, c) \in [2n] \times [n]$ of every marked box satisfies
$$n + 1 - c \leq r \leq 2n + 1 - c.$$

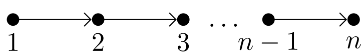
The set of all Dellac configurations for a fixed parameter n will be denoted by DC_n and its cardinality is given by the normalised median Genocchi number h_n.

Example 6.15. For $n = 3$ we list all Dellac configurations below.



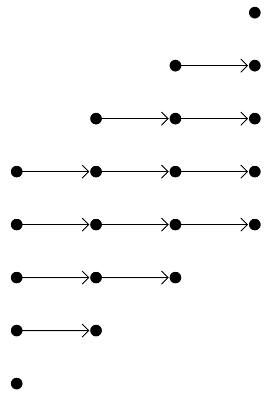
The condition (3) in the definition of Dellac configurations ensures that the marked boxes are not allowed to be on the left of the upper diagonal of marked boxes in last Dellac configuration of the example above and also not on the right of the lower diagonal of marked boxes in the same configuration. These triangles are forbidden areas for markings in all rectangles of $2n \times n$ boxes underlying a Dellac configuration.

Let Q be a linearly oriented quiver of type A_n, i.e.



and define $A := \mathbb{C}Q$ as its path algebra. The degenerate flag variety $\mathcal{F}l^a(\mathfrak{sl}_{n+1})$ is isomorphic to the quiver Grassmannian $\mathrm{Gr}^Q_{\mathbf{dim}\,A}(A \oplus A^*)$ [**20**, Proposition 2.7].

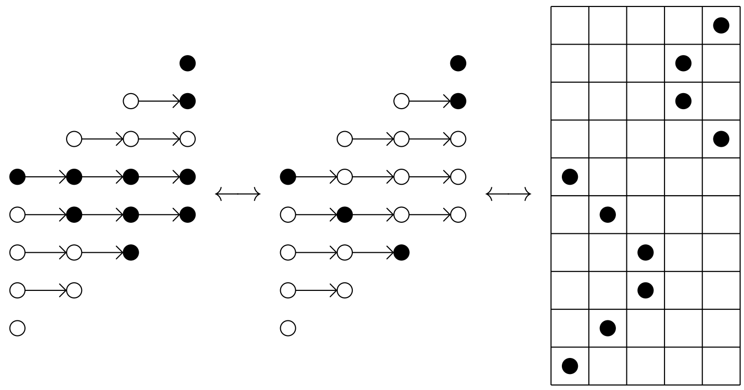
Now we want to look at the relation of cells and configurations for the degenerate flag variety $\mathcal{F}l^a(\mathfrak{sl}_5)$. In this setting, the coefficient quiver of the representation $A \oplus A^*$ is of the from



By Proposition 4.9 we obtain that the cells of the degenerate flag variety are in bijection with successor closed subquivers of this coefficient quiver which have j marked points in the j-th column. From such a subquiver we obtain a Dellac configuration by marking the boxes corresponding to the starting points of the segments of the subquiver and marking the only possible boxes in the first and the last row. If a segment contains no marked point we have to mark the box corresponding to the point to the right of the end point of this segment.

The other way around we can start with a Dellac configuration and transfer the marked points to the coefficient quiver of $A \oplus A^*$. Then we mark all necessary points to make this a successor closed subquiver. Hence the cells of $\mathcal{F}l^a(\mathfrak{sl}_{n+1})$ are in bijection with the Dellac configurations in DC_{n+1} as proven in [**29**].

Example 6.16. For the special case of $n = 4$, we show this correspondence for one successor closed subquiver in the coefficient quiver of the representation $A \oplus A^*$ of the equioriented type A quiver on four vertices.



Remark. Condition (3) in the definition of Dellac configurations is important for the dimension vector of the corresponding quiver representation. It ensures that the entries of the dimension vector of the quiver representation are increasing by one along each arrow of the quiver and that the first entry is also one. For a full flag variety and thus its degeneration, this is exactly the dimension of the vector spaces in the flag.

Moreover by Condition (2) we obtain that over each vertex of the quiver there are starting two segments of the coefficient quiver corresponding to a cell. The subsequent proposition suggests that for the degenerate affine flag variety there should be a structure parametrising the cells which is similar to the classical Dellac configurations.

PROPOSITION 6.17. In a subquiver of the coefficient quiver of M_ω which corresponds to a cell in the degenerate affine flag variety there are exactly two segments starting over each vertex.

PROOF. In Section 3.1 we have seen that orbits of quiver representations are parametrised by collections of words where each word corresponds to a indecomposable representation of the cycle. The segments in the coefficient quiver of M_ω can also be parametrised by these words. Hence cells also correspond to collections of words. Here the order of the words matters to distinguish between different cells corresponding to one orbit. The orbit structure of the variety of quiver representations is described by cutting and gluing words. Thus all cells can be obtained form the unique zero-dimensional cell by the same procedure of cutting and gluing words. Since we only move around sub-words, this procedure can not change the number of words starting over a vertex. If we move a sub-word ending at the vertex i, the remaining word starts at vertex $i+1$. Even if the resulting word is empty we keep it with the notation $w(i+1;0)$ and still count its starting point. By Proposition 6.12 we know that in the coefficient quiver of the unique zero-dimensional cell there are exactly two segments starting over each vertex. The cutting and gluing preserves this property for all other cells if we count the starting points as discussed above. ☐

6.4.2. Periodic Dellac Configurations. In the rest of this section we introduce the affine Dellac configurations to suit the cell structure of the degenerate affine flag variety. We start with the approximation

$$\mathcal{F}l_1^a\big(\widehat{\mathfrak{gl}}_n\big) \cong \mathrm{Gr}_{\mathbf{e}_1}^{\Delta_n}\big(M_1\big)$$

where

$$M_1 := \bigoplus_{i\in\mathbb{Z}_n} U_i(n)\otimes\mathbb{C}^2.$$

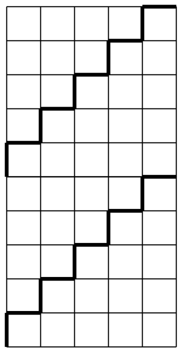
The coefficient quiver of M_1 contains $2n$ segments and each of them has length n. Thus we need at least $2n \times n$ boxes to describe the position of the starting points of subsegments. Passing form an oriented string to the oriented cycle we need some cyclic structure to describe the cells. Hence we identify the long sides of the rectangle and obtain configurations on a cylinder. Moreover we introduce a second kind of marking using white dots to denote segments which do not belong to the subquiver.

From Section 3.5.2, we know that the cells of the quiver Grassmannian above are in bijection with the set

$$\mathcal{C}_{\mathbf{e}}^{(\mathbf{d})}\big(\Delta_n, \mathrm{I}_n\big) := \Big\{\mathbf{l} := (\ell_{i,1},\ell_{i,2}) \in \bigoplus_{i\in\mathbb{Z}_n}[n]_0\times[n]_0 :\ \big(\mathbf{dim}\,U(\mathbf{l})\big)_i = n\Big\}$$

where $\mathbf{e} = (n)_{i\in\mathbb{Z}_n}$, $\mathbf{d} = (2)_{i\in\mathbb{Z}_n}$ and $N = n$ since $\omega = 1$.

We draw a separator in the rectangle of $2n \times n$ boxes to mark the end of the segments in the coefficient quiver. Hence this separator is a staircase moving diagonally around the cylinder. In the planar picture for $n = 5$ this looks like



Remark. Here we cut the cylinder such that the staircase moves from the lower left corner to the upper right corner and splits into two parts. It is possible to cut it at any other point. We have chosen this picture to empathise the similarities between cyclic and classical Dellac configurations.

Namely, the top-dimensional cell of the degenerate flag variety $\mathcal{F}l^a(\mathfrak{sl}_n)$ and one top-dimensional cell of the approximation of the degenerate affine flag variety $\mathcal{F}l^a_1(\widehat{\mathfrak{gl}}_n)$ have the same picture with this planar presentation.

Now we describe how to assign a configuration to a cell. For $k = 1$ and $\ell_{i,k} \neq 0$ we go to the i-th row, move $\ell_{i,k}$ steps to the left from the seperator and put a black dot inside the box. If $\ell_{i,k} = 0$ we move one step from the seperator to the right and fill the box with a white dot. For $k = 2$ we go to $i+n$-th row and do the same as for $k = 1$.

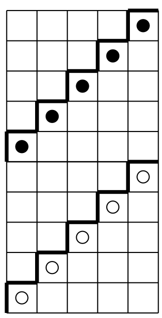
Example 6.18. For $n = 5$ and $\omega = 1$ the representation

$$U = \bigoplus_{i \in \mathbb{Z}_n} U_i(n)$$

is contained in the quiver Grassmannian

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_1}\big(M_1\big).$$

Its stratum corresponds to the tuple $(n, 0)_{i \in \mathbb{Z}_n}$ and the procedure described above assigns the configuration



This is the image of the top-dimensional cell we mentioned in the remark about the planar presentation.

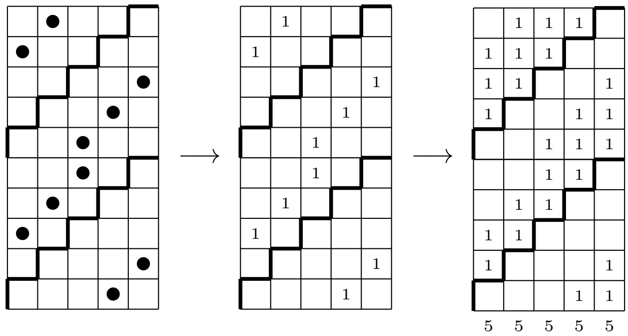
To decide whether a given configuration encodes a cell of the quiver Grassmannian, we need a tool to check if the corresponding quiver representation has the right dimension vector.

DEFINITION 6.19. Let D be a configuration in a rectangle of $2n \times n$ boxes. For each black dot we write a one into its box and all boxes on the right until we reach the diagonal separator. The **weight** of a column is the sum of its entries and the column vector with the column weights as entries is called **weight vector**.

We compute the weight vector for a cell in the ongoing example of this subsection. Take the tuple

$$\Big((3,2),(3,2),(3,2),(3,2),(3,2)\Big)$$

which corresponds to the subsequent configuration



with computation of the weight vector $(5,5,5,5,5)$.

DEFINITION 6.20. A **cyclic Dellac configuration** $\tilde{D}$ consists of $2n$ black and white dots in a rectangle of $2n \times n$ boxes such that:

(1) each row contains exactly one dot,
(2) each column contains exactly two dots,
(3) the weight of each column is n.

By $\tilde{DC}_n$ we denote the set of all cyclic Dellac configurations. These configurations provide a combinatoric description of the cells in the quiver Grassmannian which is isomorphic to the smallest non-trivial approximation of the degenerate affine flag variety.

LEMMA 6.21. *The set of cyclic Dellac configurations $\tilde{DC}_n$ is in bijection with the set*

$$\mathcal{C}_{\mathbf{e}}^{(\mathbf{d})}\big(\Delta_n, \mathrm{I}_n\big) := \Big\{\mathbf{l} := (\ell_{i,1}, \ell_{i,2}) \in \bigoplus_{i\in\mathbb{Z}_n} [n]_0 \times [n]_0 : \ \big(\mathbf{dim}\, U(\mathbf{l})\big)_i = n\Big\}.$$

In particular, the set $\tilde{DC}_n$ parametrises the cells of the quiver Grassmannian

$$\mathrm{Gr}_{\mathbf{e}_1}^{\Delta_n}\big(M_1\big).$$

PROOF. Given a tuple $\mathbf{l} \in \mathcal{C}_{\mathbf{e}}^{(\mathbf{d})}$, its entry $\ell_{i,k}$ parametrises the representation $U(i;\ell_{i,k})$ which corresponds to the word

$$w(i;\ell_{i,k}) := i-\ell_{i,k}+1 \quad i-\ell_{i,k}+2 \quad \dots \quad i-2 \quad i-1 \quad i$$

and the entries in its dimension vector are given by

$$\big(\mathbf{dim}\, U(i;\ell_{i,k})\big)_j = \begin{cases} 1 & \text{if } j \in w(i;\ell_{i,k}) \\ 0 & \text{otherwise} \end{cases}.$$

From the procedure to compute the weight of a configuration we obtain this dimension vector as the vector in the $i-th$ row of the configuration. Accordingly the dimension vector $\mathbf{dim}\, U(\mathbf{l})$ and the wight vector of the corresponding configuration are the same. This proves that all configurations in the image of $\mathcal{C}_{\mathbf{e}}^{(\mathbf{d})}$ satisfy property (3) in the definition of affine Dellac configurations.

The rows of the configuration are in one to one correspondence with segments in the coefficient quiver of M_ω and in each segment there is at most one subsegment describing a successor closed subquiver. Hence there is a black dot in the i-th row if and only if there is a subsegment in the i-th segment of the coefficient quiver of M_ω. By definition of the map we put a white dot in the i-th row if there is no subsegment in the i-th segment of the coefficient quiver. So Property (1) is satisfied since there is exactly one dot in every row of a configuration in the image of $\mathcal{C}_{\mathbf{e}}^{(\mathbf{d})}$.

The starting points of the segments of a successor closed subquiver in the coefficient quiver of M_ω are in one to one correspondence with the marked boxes of a configuration. Hence Property (2) is satisfied by Proposition 6.17.

For $\omega = 1$ there are no distinct tuples with the same positions of the starting points of the subsegments in the coefficient quiver. Since the configurations capture the position of the starting points of the segments, it is clear that the map from $\mathcal{C}_{\mathbf{e}}^{(\mathbf{d})}$ to the configurations is injective. Moreover, the configurations in the image are cyclic Dellac configurations since they satisfy Property (1), (2) and (3).

It remains two show that every cyclic Dellac configuration arises as image of a cell. Given a configuration $\tilde{D}$, we can recover the tuple $\mathbf{l}$ which is mapped to $\tilde{D}$ following the steps in the computation of the weight. We fill the boxes with 1's as in the definition of the weight. Define $\ell_{i,1}$ as the sum over the i-th row and $\ell_{i,2}$ as the sum over the $i+n$-th row.

The representation described by this tuple has the right dimension vector since the configuration $\tilde{D}$ satisfies Property (3). It embeds into M_1 because the tuple describes a successor closed subquiver in the coefficient quiver of M_1. Hence the preimage of each cyclic Dellac configuration is a tuple in the set $\mathcal{C}_{\mathbf{e}}^{(\mathbf{d})}$. □

For $\omega \geq 2$ it is not sufficient to distinguish between black and white dots. Instead we take numbers k between zero and ω to label the boxes. We generalise the notion of weights by writing k's in the boxes where we wrote 1's following the original definition. In the other boxes we write $\max\{k-1, 0\}$. For $\omega = 1$ this yields the same weight vector as the original definition.

DEFINITION 6.22. An **affine Dellac configuration** $\widehat{D}$ to the parameter $\omega \in \mathbb{N}$ consists of $2n$ numbers form zero to ω in a rectangle of $2n \times n$ boxes such that:

(1) each row contains exactly one number,
(2) each column contains exactly two numbers,
(3) the weight of each column is ωn.

The subsequent proposition is a direct consequence of the parametrisation of the cells by the length of the subsegments in the coefficient quiver.

PROPOSITION 6.23. The weight vector of an affine Dellac configuration is equal to the dimension vector of a representative for the corresponding cell in the quiver Grassmannian.

PROOF. Starting with a tuple parametrising a cell of the approximation we write $\lceil \ell_{i,k} \rceil$ in the corresponding box of the affine Dellac configuration. In the computation of the weight vector we fill the other boxes in this row which are on the right of the starting box and on the left of the separator with the same number. The other boxes of the line are filled with the number $\lfloor \ell_{i,k} \rfloor$. If we view the numbers in this row as a vector this equals the dimension vector of the indecomposable representation $U(i; \ell_{i,k})$. This is exactly the summand of $U(\mathbf{l})$ which is parametrised by $\ell_{i,k}$. Hence the sum of all row vectors computed from the $\ell_{i,k}$'s in this way equals the dimension vector of the representation $U(\mathbf{l})$. $\square$

Recall that the approximation of the degenerate affine flag variety

$$\mathcal{F}l^a_\omega(\widehat{\mathfrak{gl}}_n)$$

is in bijection with the quiver Grassmannian

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega)$$

where

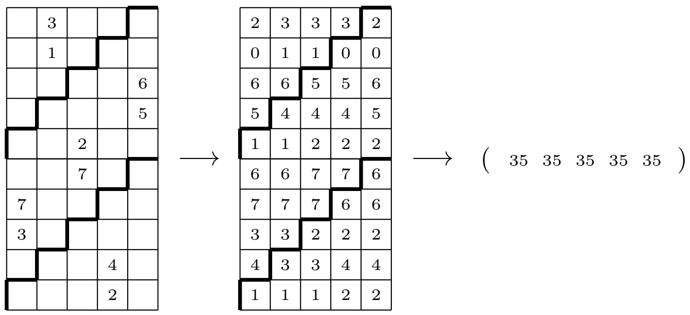
$$X_\omega = Y_\omega = \bigoplus_{i \in \mathbb{Z}_n} U_i(\omega n)$$

and $\mathbf{e}_\omega := \mathbf{dim}\, X_\omega = (\omega n)_{i \in \mathbb{Z}_n}$.

EXAMPLE 6.24. The tuple $((13, 32), (2, 33), (28, 12), (22, 18), (8, 7))$ describes a cell in

$$\mathcal{F}l^a_7(\widehat{\mathfrak{gl}}_5)$$

and the weight vector of the corresponding configuration computes as



THEOREM 6.25. *The set $\widehat{DC}_n(\omega)$ containing affine Dellac configurations to the parameter $\omega \in \mathbb{N}$ is in bijection with the cells of the approximation*

$$\mathcal{F}l^a_\omega(\widehat{\mathfrak{gl}}_n)$$

of the degenerate affine flag variety.

PROOF. If $\omega = 1$ this follows from Lemma 6.21 since the approximation is provided by the quiver Grassmannian in the lemma. For $\omega \geq 2$ let $\hat{D}$ be a Dellac configuration in the set $\widehat{DC}_n(\omega)$. By the steps in the computation of the weight vector it is also possible to recover the parameters $\ell_{i,k}$ by computing the row sums instead of the column sums. By Proposition 6.23 we know that the resulting quiver representation $U(\mathbf{l})$ has the right dimension vector. For each $i \in \mathbb{Z}_n$ there are exactly two boundaries behind the boxes in the i-th column. Hence the tuple $\mathbf{l}$ can parametrise at most two direct summands of $U(\mathbf{l})$ ending over the i-th vertex of the cycle such that there exists a segment-wise embedding of $U(\mathbf{l})$ into the representation M_ω.

Given a tuple $\mathbf{l}$ in the set

$$\Big\{\mathbf{l} := (\ell_{i,1}, \ell_{i,2}) \in \bigoplus_{i\in\mathbb{Z}_n} [\omega n]_0 \times [\omega n]_0 : \ \mathbf{dim}\, U(\mathbf{l}) = \mathbf{e}_\omega \Big\}$$

we know that the corresponding configuration satisfies Propety (1) in the definition of affine Dellac configurations by construction. In Proposition 6.23 we have checked that Property (3) is satisfied. By Proposition 6.17 we know that every successor closed subquiver corresponding to a cell in the degenerate affine flag variety has two starting points of segments over each $i \in \mathbb{Z}_n$. The labelled boxes capture the position of the starting points such that Property (2) is satisfied as well.

Two tuples with the same affine Dellac configuration as image have to be equal since the configuration allows to recover the value of each entry in the tuple as described above. □

There is a relation between cyclic and affine Dellac configurations. Clearly it is possible to construct a cyclic Dellac configuration from an affine one but it is also possible to extend cyclic configurations in order to be affine.

Remark. For any $\omega \geq 1$ we can obtain the set of affine Dellac configurations $\widehat{DC}_n(\omega)$ from the set of cyclic Dellac configurations $\widetilde{DC}_n$. Given a cyclic Dellac configuration $\tilde{D}$ we replace the black dots by 1's and the white dots by 0's. In every column we are allowed to add a number between zero and ω to the entry if the entry is zero. Otherwise we are allowed to add a number between zero and $\omega - 1$. The resulting configuration is contained in $\widehat{DC}_n(\omega)$ if the sum of the numbers we add equals $(\omega - 1)n$.

Proof. The entries of the wight vector of the configuration $\tilde{D}$ are equal to n. Adding k to the entry in the i-th line adds k to every entry of the weight vector and this is independent of the row index i. If the sum of numbers we add equals $(\omega - 1)n$ the entries in the weight vector of the new configuration are equal to ωn. Accordingly the configuration is contained in the set $\widehat{DC}_n(\omega)$. □

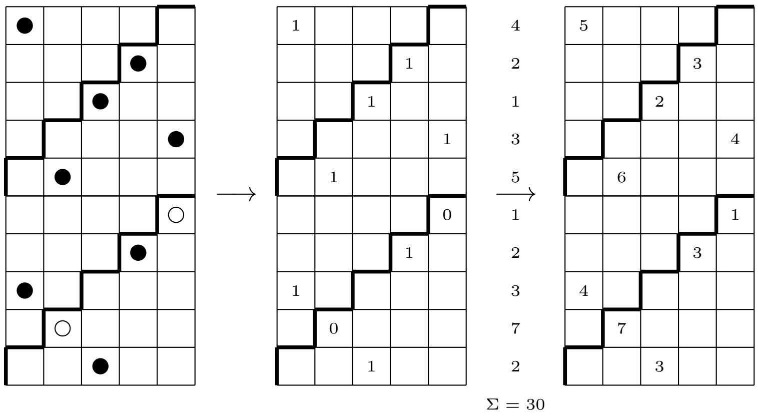
Corollary 6.26. The cardinalities of the sets of affine and cyclic Dellac configurations satisfy the relation

$$\#\widehat{DC}_n(\omega) \leq \binom{(\omega-1)n + 2n - 1}{2n - 1} \#\widetilde{DC}_n$$

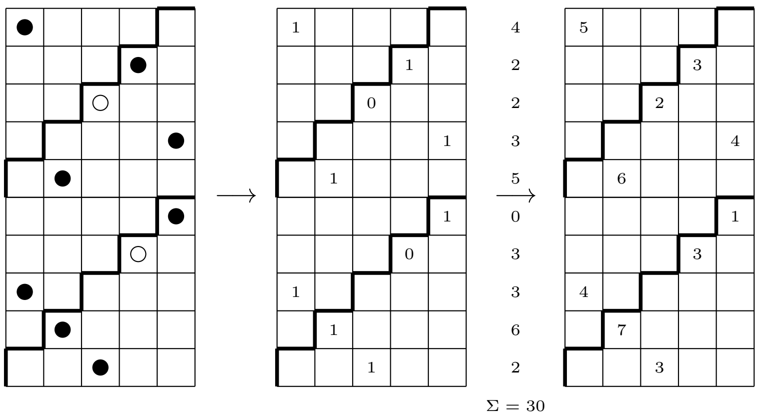
Proof. The coefficient on the right hand side counts the possibilities to split the number $(\omega - 1)n$ into at most $2n$ parts where two splittings are distinguished if their parts have a different order. Every splitting of $(\omega - 1)n$ in the sense of the remark above is covered by this. □

The procedure to obtain affine Dellac configurations from cyclic Dellac configurations is not unique. It is possible to obtain the same affine Dellac configuration from different cyclic Dellac configurations.

Example 6.27. For $n = 5$ and $\omega = 7$ we construct an affine configuration in $\widehat{DC}_5(7)$ from two different cyclic configurations in $\widetilde{DC}_5$. Starting with a cyclic Dellac configuration we have to add numbers with a total ammount of 30 to the entries of the cyclic Dellac configuration.



The following steps describe one other possible construction of the same configuration.



Any other configuration with a different arrangement of black and white dots on the diagonal would also provide a suitable starting point to construct the same affine Dellac configuration. If an affine Dellac configuration can be constructed from a cyclic Dellac configuration which has no white dots, this is the only possibility to construct this configuration.

6.4.3. The Length of a Configuration. Property (3) in the definition of affine Dellac configurations is necessary to get the right dimension vector for the corresponding quiver representation. Nevertheless it is desirable to replace this condition by something which is easier to check.

For classical Dellac configurations this condition is simply a restriction on the areas where the dots are allowed. Unfortunately, the introduction of weights requires that we at least have to control the sum of all weights given to the boxes. The new tool to distinguish configurations is called length and will be introduced below.

Let $\widetilde{D} \in \widetilde{DC}_n$ be a cyclic Dellac configuration. In every row of the configuration $\widetilde{D}$ we count the steps which a black dot moved from the separator to its actual position. This number will be denoted by p_j for $j \in \mathbb{Z}_{2n}$ because it encodes the information about the position of the dot in the j-th row. For white dots we set $p_j = 0$. This is compatible with the definition for black dots since the white dots can not have moved away from their starting position. For black dots we have $p_j \geq 1$ and if a black dot has returned to the position of a white dot we have $p_j = n$.

Definition 6.28. The **length** of the configuration $\widetilde{D} \in \widetilde{DC}_n$ is defined as

$$\mathrm{len}(\widetilde{D}) := \sum_{j \in \mathbb{Z}_{2n}} p_j.$$

We say that the configuration $\widetilde{D}$ satisfies Property $(3)'$ if $\mathrm{len}(\widetilde{D}) = n^2$.

Proposition 6.29. Any cyclic Dellac configuration $\widetilde{D} \in \widetilde{DC}_n$ satisfies Property $(3)'$.

Proof. In the computation of the weight vector the number of 1's we write into the j-th row equals the number p_j. The sum of the entries in the weight vector of $\widetilde{D}$ equals n^2 since $\widetilde{D}$ satisfies Property (3). For the weight vector we compute column sums and the length vector

$$\mathbf{len}(\widetilde{D}) := (p_j)_{j \in \mathbb{Z}_{2n}}$$

contains the row sums. Hence the sum over the entries in both vectors has to be the same, i.e.

$$n^2 = \sum_{j \in \mathbb{Z}_{2n}} p_j = \mathrm{len}(\widetilde{D}).$$

□

The proof of the other direction requires a bit more work. First we describe a different approach to define affine Dellac configurations which is based on Property $(3)'$. For that we need the subsequent notion of configurations.

Definition 6.30. In a rectangle of $2n \times n$ boxes a **periodic Dellac configuration** $\overline{D}$ consists of $2n$ marked boxes such that:

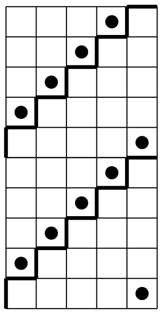
(1) each row contains exactly one marked box,
(2) each column contains exactly two marked boxes.

Every Dellac configuration is a periodic Dellac configuration. If we split the cylinder of boxes at a different point, we obtain a periodic Dellac configuration again. The set of all periodic Dellac configurations in $2n \times n$ boxes is denoted by $\overline{DC}_n$.

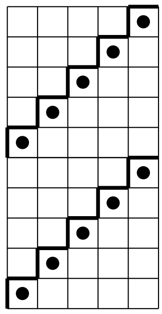
Remark. We can obtain affine Dellac configurations from periodic Dellac configurations by replacing the dots by numbers from zero to ω such that Property (3) is satisfied.

Here the entry zero is only allowed on the marked diagonal. But this restriction is implied by Property (3). This process to obtain affine Dellac configurations is similar to the one describe above for cyclic Dellac configurations.

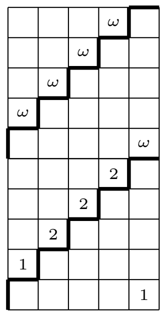
The main difference is that now the underlying periodic Dellac configuration of every affine Dellac configuration is unique. But the weight vector of the underlying periodic Dellac configuration is not known. Its entries can vary between 2 and $2n$. These extremal entries of the weight vector are obtained with configurations of the form



and



Accordingly it is not clear which total amount of numbers we have to insert into a periodic Dellac configuration in order to make it affine for the parameter ω. From the left configuration above we know that at least inserting a total amount of $\omega n + 2n - 2$ is sufficient since the configuration

$\widehat{D} :=$ 

has the weight vector

$$\mathbf{wt}(\widehat{D}) = \big((\omega-1)n + 2 + (n-2)\big)_{i\in\mathbb{Z}_n} = \big(\omega n\big)_{i\in\mathbb{Z}_n}.$$

For arbitrary n this configuration contains n entries which are equal to ω, $n-2$ entries equal to 2 and 2 times the entry one. As above we obtain an upper bound on the cardinality of the set of affine Dellac configurations, i.e.

$$\#\widehat{DC}_n(\omega) \leq \binom{(\omega+1)n + 2n}{2n} \#\overline{DC}_n.$$

For an affine Dellac configuration $\widehat{D} \in \widehat{DC}_n(\omega)$ we compute its length as follows. Let k_j be the entry in the j-th row of the configuration. We determine the the position of this entry as defined above for the cyclic configurations and denote it by $p_j := p(k_j)$. Remember that the position of a white dot is zero. Hence the same convention leads to $p(0) = 0$. The winding number r_j of the j-th row is defined as

$$r_j := \max\big\{k_j - 1, 0\big\}$$

and counts the full rounds around cycle which can be cut out of the corresponding segment in the coefficient quiver without sending it to zero. For a cyclic Dellac configuration these numbers are all equal to zero since $\omega = 1$ and hence every segment goes around the cycle at most once. Accordingly

$$\operatorname{len}(\widehat{D}) := n \sum_{j\in\mathbb{Z}_{2n}} r_j + \sum_{j\in\mathbb{Z}_{2n}} p_j$$

generalises the notion of length to affine Dellac configurations. The generalisation of Property (3)′ is given by

$$\operatorname{len}(\widehat{D}) = n \sum_{j\in\mathbb{Z}_{2n}} r_j + \sum_{j\in\mathbb{Z}_{2n}} p_j = \omega n^2.$$

The subsequent observation is a first step for the other direction in the proof of the equivalence of Property (3) and Property (3)′.

Proposition 6.31. Let $\widehat{D} \in \widehat{DC}_n(\omega)$ be an affine Dellac configuration such that $k_j \geq 1$ for all $j \in \mathbb{Z}_{2n}$. The positions p_j for $j \in \mathbb{Z}_{2n}$ describe a periodic Dellac configuration $\overline{D}_{\mathbf{p}} \in \overline{DC}_n$ with weight vector

$$\mathbf{wt}(\overline{D}) = \Big(\frac{1}{n} \sum_{j\in\mathbb{Z}_{2n}} p_j\Big).$$

Proof. The entries of an affine Dellac configuration satisfy Property (1), Property (2) and Property (3). We replace every entry by a 1 or equivalently a black dot. The resulting configuration still satisfies Property (1) and Property (2). Hence it is periodic. The weight of this configuration is given by

$$\operatorname{wt}\big(\overline{D}_{\mathbf{p}}\big) = \sum_{j\in\mathbb{Z}_{2n}} p_j.$$

For the configuration $\widetilde{D}$, it is a homogeneous transformation of the weight vector to decrease an entry of the configuration by some integer. Namely, decreasing any entry $k_j \geq 1$ of the configuration by a number

$$q \leq r_j = k_j - 1$$

decreases every entry of the weight vector by q. Accordingly the weight vector of $\overline{D}_{\mathbf{p}}$ is homogeneous and its entries are given by

$$\frac{\operatorname{wt}\big(\overline{D}_{\mathbf{p}}\big)}{n}.$$

In particular this number is an integer. □

The following proposition is required for the generalisation of the proof that Property (3) implies Property (3)′.

Proposition 6.32. Let $\widehat{D} \in \widehat{DC}_n(\omega)$ be an affine Dellac configuration. Then

$$\operatorname{wt}\big(\,\widehat{D}\,\big) = \operatorname{len}\big(\,\widehat{D}\,\big).$$

Proof. The sum of the entries in the j-th row of the diagram in the computation of the weight vector equals

$$\ell_j := n \max\big\{k_j - 1, 0\big\} + p_j.$$

By construction, this is the length of the segment in the coefficient quiver which is parametrised by the j-th row of the configuration. Summation over $j \in \mathbb{Z}_{2n}$ yields

$$\sum_{j\in\mathbb{Z}_{2n}} \big(n \max\big\{k_j - 1, 0\big\} + p_j\big) = n \sum_{j\in\mathbb{Z}_{2n}} \max\big\{k_j - 1, 0\big\} + \sum_{j\in\mathbb{Z}_{2n}} p_j = \operatorname{len}\big(\,\widehat{D}\,\big).$$

To compute the weight of this configuration we first compute the weight vector, i.e the column sums over the diagram and then we sum over the entries of this vector. Hence the weight and the length of the configuration $\widehat{D}$ are both obtained as the sum over all entries in the diagram which arises in the computation of the weight vector. □

LEMMA 6.33. *A periodic Dellac configuration* $\overline{D}(\mathbf{k})$ *with entries weighted by*

$$\mathbf{k} = \big(k_j\big)_{j\in\mathbb{Z}_{2n}} \in \bigoplus_{j\in\mathbb{Z}_{2n}} [\omega]_0$$

satisfies Property (3) *if and only if it satisfies Property* (3)′.

PROOF. The Dellac configuration where we write k_j instead of a dot is denoted by $\overline{D}(\mathbf{k})$. Assume that this configuration satisfies Property (3), i.e.

$$\mathbf{wt}\big(\,\overline{D}(\mathbf{k})\,\big) = \big(\omega n\big)_{i\in\mathbb{Z}_n}.$$

We obtain the weight

$$\mathrm{wt}\big(\overline{D}(\mathbf{k})\,\big) = \sum_{i\in\mathbb{Z}_n} \omega n = \omega n^2.$$

By Proposition 6.32 this is equal to $\mathrm{len}\big(\overline{D}(\mathbf{k})\big)$ such that Property (3)′ is satisfied.

Conversely assume that $\overline{D}(\mathbf{k})$ satisfies Property (3)′, i.e.

$$\mathrm{len}\big(\,\overline{D}(\mathbf{k})\,\big) = \omega n^2.$$

First we consider the case that $k_j \geq 1$ for all $j \in \mathbb{Z}_{2n}$. Then by Proposition 6.31, the underlying configuration $\overline{D}$ has the weight vector

$$\mathbf{wt}\big(\,\overline{D}\,\big) = \Big(\,\frac{1}{n}\sum_{j\in\mathbb{Z}_{2n}} p_j\,\Big)_{i\in\mathbb{Z}_n}.$$

In this setting, we can view the transformation from $\overline{D}$ to $\overline{D}(\mathbf{k})$ as adding $k_j - 1$ to the entry in the j-th row of the configuration $\overline{D}$. This operation increases every entry of the weight vector by $k_j - 1$ such that we obtain

$$\begin{aligned}
\mathbf{wt}\big(\,\overline{D}(\mathbf{k})\,\big) &= \Big(\,\sum_{j\in\mathbb{Z}_{2n}} \big(k_j - 1\big) + \frac{1}{n}\sum_{j\in\mathbb{Z}_{2n}} p_j\,\Big)_{i\in\mathbb{Z}_n} \\
&= \Big(\,\sum_{j\in\mathbb{Z}_{2n}} \big(\max\{k_j-1,0\}\big) + \frac{1}{n}\sum_{j\in\mathbb{Z}_{2n}} p_j\,\Big)_{i\in\mathbb{Z}_n} \\
&= \Big(\,\frac{1}{n}\,\mathrm{len}\big(\,\overline{D}(\mathbf{k})\,\big)\,\Big)_{i\in\mathbb{Z}_n} = \big(\,\omega n\,\big)_{i\in\mathbb{Z}_n}.
\end{aligned}$$

Accordingly the configuration $\overline{D}(\mathbf{k})$ satisfies Property (3).

Now assume that we have $k_j = 0$ for some $j \in \mathbb{Z}_{2n}$. For the underlying configuration $\overline{D}$ we have $p_j = p_j(\overline{D}) = n$ but for the configuration $\overline{D}(\mathbf{k})$ we get $p(k_j) = 0$. The weight vector of the underlying configuration is given by

$$\mathbf{wt}\big(\,\overline{D}\,\big) = \Big(\,\frac{1}{n}\sum_{j\in\mathbb{Z}_{2n}} p_j\big(\,\overline{D}\,\big)\,\Big)_{i\in\mathbb{Z}_n}.$$

For the summands we obtain the equality

$$\frac{p_j\big(\,\overline{D}\,\big)}{n} + \min\{0, k_j - 1\} = \frac{p(k_j)}{n}.$$

Replacing a black dot in the configuration $\overline{D}$ by the number k_j increases the entries of the weight vector by $k_j - 1$ if $k_j \geq 1$ and it decreases the entries by 1 if $k_j = 0$. Thus the weight vector of $\overline{D}(\mathbf{k})$ computes as

$$\begin{aligned}
\mathbf{wt}\big(\overline{D}(\mathbf{k})\big) &= \Big(\sum_{j\in\mathbb{Z}_{2n}} \big(k_j - 1\big) + \frac{1}{n} \sum_{j\in\mathbb{Z}_{2n}} p_j\big(\overline{D}\big) \Big)_{i\in\mathbb{Z}_n} \\
&= \Big(\sum_{j\in\mathbb{Z}_{2n}} \big(k_j - 1\big) + \frac{1}{n} \sum_{j\in\mathbb{Z}_{2n}} \big(p(k_j) - n\cdot\min\{0, k_j - 1\}\big) \Big)_{i\in\mathbb{Z}_n} \\
&= \Big(\sum_{j\in\mathbb{Z}_{2n}} \big((k_j - 1) - \min\{0, k_j - 1\}\big) + \frac{1}{n} \sum_{j\in\mathbb{Z}_{2n}} p(k_j) \Big)_{i\in\mathbb{Z}_n} \\
&= \Big(\sum_{j\in\mathbb{Z}_{2n}} \big(\max\{k_j - 1, 0\}\big) + \frac{1}{n} \sum_{j\in\mathbb{Z}_{2n}} p(k_j) \Big)_{i\in\mathbb{Z}_n} = \big(\omega n\big)_{i\in\mathbb{Z}_n}.
\end{aligned}$$

Here the last equality follows since $\overline{D}(\mathbf{k})$ satisfies Property $(3)'$. □

This proves the subsequent characterisation of affine Dellac configurations.

THEOREM 6.34. *An* ***affine Dellac configuration*** $\widehat{D}$ *to the parameter* $\omega \in \mathbb{N}$ *consists of* $2n$ *numbers form zero to* ω *in a rectangle of* $2n \times n$ *boxes such that:*

(i) *each row contains exactly one number,*

(ii) *each column contains exactly two numbers,*

(iii) *the length of* $\widehat{D}$ *is given by* ωn^2*.*

With this alternative characterisation it is easier to construct affine Dellac configurations than with the original definition. Here we only need a periodic Dellac configuration as basis. Then we can introduce any tuple of weights to the dots which satisfies Condition *(iii)* of the above theorem.

Originally we had to check Property (3) in the definition of cyclic Dellac configurations and then could add tuples summing up to $(\omega - 1)n$. So we reduced the number of conditions by one.

Moreover we got rid of the graphical step where we had to write the dimension vectors in the rows of the configuration. This also makes it easier to decide whether a given configuration can be an affine Dellac configuration since we do not have to draw this diagram any more.

The procedure of this decision will be as follows. First check Property (1) and Property (2). If they are satisfied compute the sum

$$\sum_{j\in\mathbb{Z}_{2n}} \max\{k_j - 1, 0\}$$

which has to be between $\omega n - 2$ and $(\omega - 2)n$. Finally add

$$\frac{1}{n} \sum_{j\in\mathbb{Z}_{2n}} p(k_j)$$

to this number. The result has to equal ωn if the configuration is an affine Dellac configuration to the parameter ω.

6.5. Geometric Properties

In this section we apply the theory of quiver Grassmannians for the equioriented cycle as introduced in the previous chapters of this thesis to the approximations of the degenerate affine flag variety. This allows to derive properties of their geometry from the study of the corresponding quiver Grassmannians.

THEOREM 6.35. *For $\omega \in \mathbb{N}$, the approximation $\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ of the Feigin-degenerate affine flag variety satisfies:*

(1) It is a projective variety of dimension ωn^2.
(2) It admits a cellular decomposition.
(3) There is a bijection between the cells and affine Dellac configurations to the parameter ω.

The irreducible components of the finite dimensional approximation of the Feigin-degenerate affine flag variety satisfy:

(4) They are equidimensional.
(5) They have rational singularities and are normal, Cohen-Macaulay.
(6) There is a bijection between the irreducible components and grand Motzkin paths of length n.

PROOF. In Theorem 6.4 we have established an isomorphism between the finite dimensional approximations of the flag variety and quiver Grassmannians for the equioriented cycle. In Corollary 6.8 we have computed the dimension of these quiver Grassmannians which are projective varieties. We have proven in Theorem 6.25 that the cells of the approximations are parametrised by affine Delllac configurations to the parameter ω. The irreducible components where examined in Lemma 6.10. From the shape of the quiver representations X_ω and Y_ω as introduced in Theorem 6.4 it follows that we can apply Lemma 3.24 proving the rationality of the singularities to the irreducible components of the quiver Grassmannian providing the approximation. □

6.6. The Non-Degenerate Affine Flag Variety

In this section we examine the structure of the non-degenerate affine flag variety. Its finite approximations can be identified with quiver Grassmannians for the equioriented cycle similarly to the case of the affine flag variety.

THEOREM 6.36. *Let $\omega \in \mathbb{N}$ be given, take the nilpotent quiver representation $M_\omega = \bigoplus_{i\in\mathbb{Z}_n} U_i(2\omega n)$ and define the dimension vector $\mathbf{e}_\omega := 1/2 \cdot \mathbf{dim}\, M_\omega$. The corresponding quiver Grassmannian is isomorphic to the approximation of the affine flag variety, i.e.*

$$\mathcal{F}l_\omega\big(\widehat{\mathfrak{gl}}_n\big) \cong \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(M_\omega).$$

The proof of this theorem works in the same way as for the degenerate affine flag variety. First we have to interpret the coefficient quiver of M_ω suiting the maps between the Sato Grassmannians.

PROPOSITION 6.37. The quiver representation M_ω is isomorphic to the quiver representation

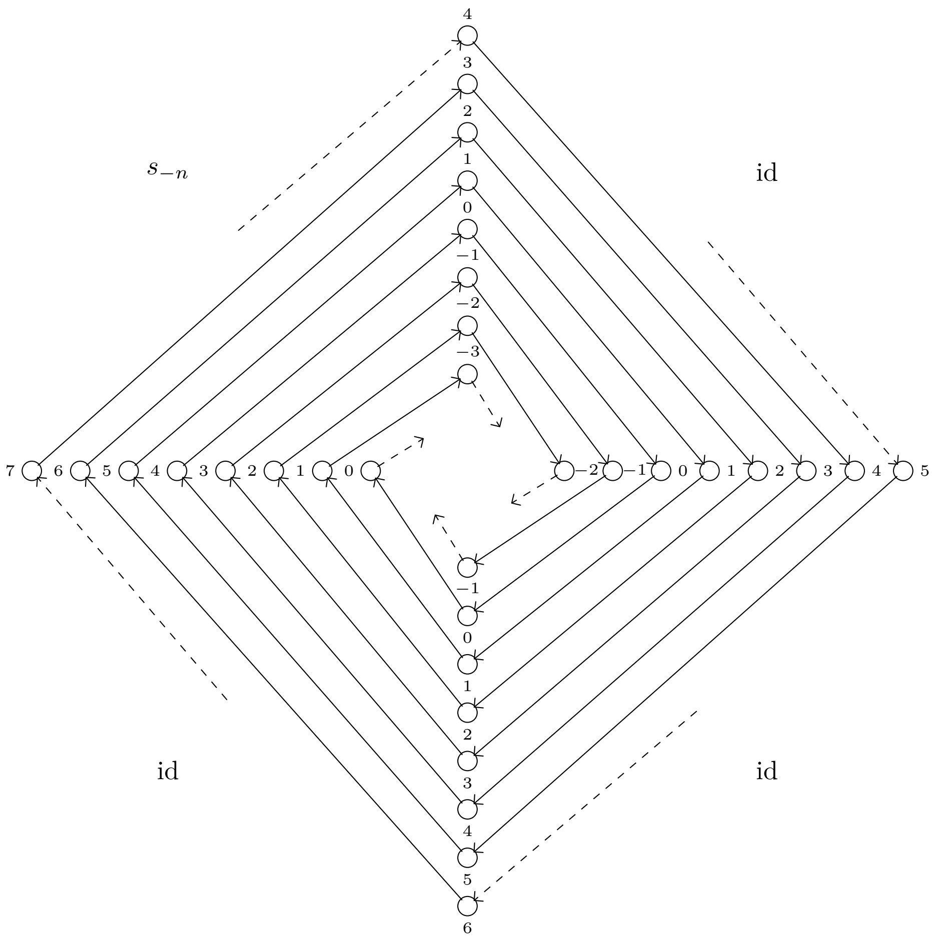
$$M^0_\omega := \big(M_{\alpha_i} := s_1\big)_{i\in\mathbb{Z}_n}$$

where s_1 is the index shift by one.

PROOF. The vertices of Δ_n are in bijection with the set $\mathbb{Z}_n$ and we choose the representatives $0, 1, \ldots, n-1$. For the representation M_ω the vectorspace over each vertex $i \in \mathbb{Z}_n$ has dimension $2\omega n$. For the arrangement of the segments in the coefficient quiver corresponding to the summands $U_i(2\omega n)$ of M_ω as in Section 4.4 we obtain the maps $s_1 : \mathbb{C}^{2\omega n} \to \mathbb{C}^{2\omega n}$ along the arrows of Δ_n since there is starting exactly one segment of length $2\omega n$ over each vertex. It ends in the basis vector indexed by $2\omega n$ over the vertex $i + 2\omega n - 1 = i - 1$ and along each step of the segment the index of the corresponding basis vectors increases by one. □

We change the indices of the basis vectors over the vertices $i \in \mathbb{Z}_n$ in order to match the indices of the basis vectors for the spaces in the Sato Grassmannians SGr_i. This is done in the same way as for $X_\omega \oplus Y_\omega$ in the degenerate setting since the vector spaces over the vertices of Δ_n have dimension $2\omega n$ in both cases. The coefficient quiver of the representation M^0_ω with the new labelling is show below.

EXAMPLE 6.38. Using the new labelling the coefficient quiver of M_ω for $n = 4$ and $\omega = 1$ is given by



Again the dashed arrows in the picture indicate where the segments grow if we increase the value of ω.

REMARK. Here all summands of M_ω are injective representations of the length $2\omega n$ for the set of relations $\mathrm{I}_{2\omega n}$. Hence we can apply Theorem 2.3 in order to study the irreducible components of the approximations.

LEMMA 6.39. *For* $\omega \in \mathbb{N}$*, the irreducible components of the finite approximation* $\mathcal{F}l_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ *are of dimension*

$$2\omega\lfloor n/2\rfloor\lceil n/2\rceil,$$

normal, Cohen-Macaulay, have rational singularities and their number is given by

$$\frac{n!}{\lfloor n/2\rfloor!\lfloor n/2\rfloor!}.$$

PROOF. From Theorem 3.5 by G. Kempken we know that the orbit closures in the variety of quiver representations have rational singularities. In both cases all summands can be viewed as bounded injective representations such that we can apply Theorem 2.3 to transport the rational singularities from the variety of quiver representations to the quiver Grassmannian. The other two properties are a direct consequence of Theorem 3.4 by G. Kempf. It remains to compute the dimension and number of irreducible components from the orbit structure of the variety of quiver representations. In this computation we have to distinguish between n even and n odd.

For n even one can apply Lemma 3.22 in order to compute the dimension of the approximations. With

$$x_i = 1,\ x_{n/2+i} = 0 \quad \text{for } i \in \mathbb{Z}_{n/2}$$

and

$$y_{n/2+i-1} = 1,\ y_{i-1} = 0 \quad \text{for } i \in \mathbb{Z}_{n/2}$$

the dimension of the approximation computes as

$$\dim \mathcal{F}l_\omega\big(\widehat{\mathfrak{gl}}_n\big) = \frac{\omega}{2}n^2.$$

By definition we have $y_i + x_{i+1} = 1$ for all $i \in \mathbb{Z}_n$. Hence we obtain by Lemma 3.23 that the irreducible components of the approximation are parametrised by the set

$$C_{n/2}(\mathbf{1}) := \Big\{\mathbf{p} \in \mathbb{Z}^n_{\geq 0} : p_i \leq 1 \text{ for all } i \in \mathbb{Z}_n, \sum_{i\in\mathbb{Z}_n} p_i = n/2\Big\}$$

which is in bijection with the set of $n/2$-element subsets of the set $[n]$. Accordingly the number of irreducible components equals $\binom{n}{n/2}$ which matches the claimed number for even n.

For n odd we can not apply the results from Section 3.3 but nevertheless we can use the methods from Chapter 3 to examine the quiver Grassmannians providing the approximations of the non-degenerate affine flag variety. Following the steps in the proof of Proposition 3.21 we arrive at the subsequent representatives for the highest dimensional orbits in the variety of quiver representations

$$U := U_{i_0}(\omega n) \oplus \bigoplus_{i\in I} U_i(2\omega n).$$

where I is a subset of $\mathbb{Z}_n$ with $\lfloor n/2\rfloor$ many pairwise distinct elements. These representatives also parametrise the irreducible components of the quiver Grassmannian and the dimension of the strata computes as

$$\begin{aligned}\dim \mathcal{S}_U &= \dim \operatorname{Hom}_{\Delta_n}\big(U, M_\omega\big) - \dim \operatorname{Hom}_{\Delta_n}\big(U,U\big)\\ &= 2\omega\lfloor n/2\rfloor n + \omega n - \big(2\omega\lfloor n/2\rfloor\lfloor n/2\rfloor + 2\omega\lfloor n/2\rfloor + \omega\big)\\ &= 2\omega\lfloor n/2\rfloor\big(n-\lfloor n/2\rfloor\big) + \omega n - \big(2\omega\lfloor n/2\rfloor + \omega\big)\\ &= 2\omega\lfloor n/2\rfloor\lceil n/2\rceil + \omega n - \big(\omega(n-1)+\omega\big)\\ &= 2\omega\lfloor n/2\rfloor\lceil n/2\rceil.\end{aligned}$$

There are

$$\binom{n}{\lfloor n/2\rfloor}$$

possibilities to choose the set $I \subseteq \mathbb{Z}_n$. For each choice of I we have $\lceil n/2\rceil$ possible choices for the index i_0 since I has $\lfloor n/2\rfloor$ many elements. Hence the number of irreducible components is given by

$$\lceil n/2\rceil\binom{n}{\lfloor n/2\rfloor} = \frac{\lceil n/2\rceil n!}{\lfloor n/2\rfloor!(n-\lfloor n/2\rfloor)!} = \frac{\lceil n/2\rceil n!}{\lfloor n/2\rfloor!\lceil n/2\rceil!} = \frac{n!}{\lfloor n/2\rfloor!\lfloor n/2\rfloor!}.$$

□

6.7. Linear Degenerations of Affine Flag Varieties

In this section we want to define linear degenerations of the affine flag variety following the approach of G. Cerulli Irelli, X. Fang, E. Feigin, G. Fourier and M. Reineke as introduced in [**19**]. This will generalise the degeneration of the affine flag variety which is introduced in the beginning of this chapter.

In [**19**] a flag variety of type A_n is degenerated by relaxing the inclusion of the subspace $U_i \subseteq U_{i+1}$ to the inclusion of the image via some linear maps $f_iU_i \subseteq U_{i+1}$. They show that the resulting linear degenerate flag variety only depends on the co-ranks of the maps f_i and not the maps itself.

6.7.1. Setting. Let V be an infinite dimensional vector space over the field $\mathbb{C}$ with basis vectors v_j for $j \in \mathbb{Z}$. The set $\operatorname{Hom}(V,V)$ contains all linear maps from V to V. We consider tuples of linear maps

$$f := (f_i)_{i\in\mathbb{Z}_n} \in \prod_{i\in\mathbb{Z}_n} \operatorname{Hom}(V,V) =: \operatorname{End}^{\times n}(V).$$

On the space $\operatorname{End}^{\times n}(V)$ of tuples of linear maps we have an action of the group $G := \prod_{i\in\mathbb{Z}_n}\operatorname{GL}(V)$ via base change

$$\begin{aligned} G \times \operatorname{End}^{\times n}(V) &\longrightarrow \operatorname{End}^{\times n}(V)\\ (g,f) &\longmapsto g.f\end{aligned}$$

where

$$g.f := \Big(g_1f_0g_0^{-1}, g_2f_1g_1^{-1}, \dots, g_0f_{n-1}g_{n-1}^{-1}\Big).$$

The orbit $\mathcal{O}_{\mathrm{iso}} := G.(s_1,\dots,s_1)$ consisting of tuples where every map is an isomorphism is open in $\operatorname{End}^{\times n}(V)$. This is shown as follows.

Define the finite dimensional subspace

$$V^{(\ell)} := \operatorname{span}\big(v_\ell, v_{\ell-1}, \dots, v_{-\ell+2}, v_{-\ell+1}\big).$$

This induces a finite dimensional version of the above setup where the map s_1 is nilpotent and the corresponding orbit is open by Proposition 3.9. It is open for every finite approximation $V^{(\ell)}$ and the natural embedding $V^{(\ell)} \hookrightarrow V^{(\ell+1)}$ preserves the local structure such that the ind-topology and the Zariski topology coincide.

Let $U := \big(U_i\big)_{i\in\mathbb{Z}_n}$ be a tuple of subspaces in V such that each U_i is contained in the Sato Grassmannian SGr_0, i.e.

$$U = \big(U_i\big)_{i\in\mathbb{Z}_n} \in \prod_{i\in\mathbb{Z}_n} \mathrm{SGr}_0 =: \mathrm{SGr}^{\times n}.$$

On the product of Sato Grassmannians the group G acts via translation

$$\begin{aligned} G \times \mathrm{SGr}^{\times n} &\longrightarrow \mathrm{SGr}^{\times n} \\ (g, U) &\longmapsto g.U \end{aligned}$$

where

$$g.U := \big(g_0U_0, g_1U_1, \ldots, g_{n-1}U_{n-1}\big).$$

Definition 6.40. A tuple of maps $f \in \mathrm{End}^{\times n}(V)$ and a tuple of vector spaces $U \in \mathrm{SGr}^{\times n}$ are **compatible** if

$$f_i(U_i) \subset U_{i+1} \quad \text{for all } \ i \in \mathbb{Z}_n.$$

The variety of compatible pairs is defined as

$$\mathrm{EG}^{\times n}(V) := \Big\{(f, U) \in \mathrm{End}^{\times n}(V) \times \mathrm{SGr}^{\times n} : f \text{ and } U \text{ are compatible}\Big\}.$$

Remark. The notion of compatibility generalises the definitions of the affine flag variety and the degenerate flag variety. For the tuple f where every map f_i is equal to the index shift s_{-1}, the tuples U which are compatible with the index shifts are exactly the points in the affine flag variety as shown in Theorem 6.36. In the case where every f_i equals the shifted projection $s_{-1} \circ \mathrm{pr}_1$ the compatible tuples U are in bijection with the points of the degenerate affine flag variety as shown in Theorem 6.4. For both identifications it is essential that the affine flag variety is isomorphic to the set

$$\mathcal{F}l\big(\widehat{\mathfrak{gl}}_n\big) \cong \Big\{\big(U_k\big)_{k\in\mathbb{Z}_n} \in \mathrm{SGr}^{\times n} \ : \ s_{-1}U_i \subset U_{i+1} \text{ for all } i \in \mathbb{Z}_n\Big\}$$

which follows directly from the definition of the Sato Grassmannians SGr_i.

Let π be the projection

$$\pi : \mathrm{EG}^{\times n}(V) \longrightarrow \mathrm{End}^{\times n}(V)$$

and p the projection

$$p : \mathrm{EG}^{\times n}(V) \longrightarrow \mathrm{SGr}^{\times n}.$$

The remark above suggests the subsequent generalisation of the definitions of the affine flag variety and its degeneration as given in the beginning of this chapter.

Definition 6.41. For $f \in \mathrm{End}^{\times n}(V)$ the **f-linear degenerate affine flag variety** is defined as

$$\mathcal{F}l^f\big(\widehat{\mathfrak{gl}}_n\big) := \pi^{-1}(f).$$

We call the map

$$\pi : \mathrm{EG}^{\times n}(V) \longrightarrow \mathrm{End}^{\times n}(V)$$

universal linear degeneration of the affine flag variety $\mathcal{F}l\big(\widehat{\mathfrak{gl}}_n\big)$. The subset of $\mathrm{End}^{\times n}(V)$ over which π is flat is denoted by U_{flat} and $U_{\mathrm{flat,irr}}$ is the subset of $\mathrm{End}^{\times n}(V)$ over which π is flat with irreducible fibres.

The irreducible components of the degenerate affine flag variety, as studied in the previous sections of this chapter, are in bijection with grand Motzkin paths. Thus it is not included in the flat irreducible locus $U_{\mathrm{flat,irr}}$. For all examples of the partial degenerate affine flag varieties as computed in Appendix C.1 we do not obtain any case where the finite approximations of the affine flag variety and some of its partial degenerations are equidimensional.

An endomorphism $f \in \mathrm{End}^{\times n}(V)$ is called **nilpotent** if there exists an $\ell \in \mathbb{N}$ such that the restrictions of f_i to $V^{(\ell)}$ are nilpotent in the sense of Chapter 3. The set of nilpotent endomorphism is denoted by $\mathrm{End}^{\times n}_{nil}(V)$. The G-orbits of the nilpotent endomorphisms can be studied with the methods from the thesis by G. Kempken and the corresponding degenerations of the affine flag variety admit finite approximations by quiver Grassmannians for the equioriented cycle.

6.7.2. Loci via Co-Rank Tuples. From now on we restrict us to $\mathrm{End}^{\times n}_{nil}(V)$. In this section we want to examine the fibres of the map π and classify the different types of linear degenerations which arise as fibres of π.

For any finite quiver Q an isomorphism of Q-representations $\Psi : M \to N$ yields the isomorphism

$$\mathrm{Gr}^Q_{\mathbf{e}}(M) \cong \mathrm{Gr}^Q_{\mathbf{e}}(N).$$

This holds because for any subrepresentation (U, ϕ) of M we obtain that the representation $(\Psi(U), \Psi \circ \phi \circ \Psi^{-1})$ is a subrepresentation of N.

In the same way we can show that the fibres of two endomorphisms which live in the same G-orbit are isomorphic. Again we restrict to the finite dimensional setting for some $\ell \in \mathbb{N}$ where the isomorphism of the fibres is obtained similar as for the quiver Grassmannians in the above proposition. The isomorphisms are compatible with the embeddings $V^{(\ell)} \hookrightarrow V^{(\ell+1)}$ such that they lift to the ind-varieties.

Let $f \in \mathrm{End}^{\times n}_{nil}(V)$ be a tuple of linear maps and let $N \in \mathbb{N}$ be given such that f is nilpotent where we view the f_i as maps between the finite approximations $V^{(N)}$. We assign a tuple $\mathbf{c}$ of integers to f, where

$$\mathbf{c} := (c_{i,k})_{i\in\mathbb{Z}_n, k\in\mathbb{Z}_N} \quad \text{and} \quad c_{i,k} := \mathrm{corank}\big(f_{i+k} \circ f_{i+k-1} \circ \cdots \circ f_{i+1} \circ f_i\big).$$

Here every index is viewed as a number in $\mathbb{Z}_n$ and we call $\mathbf{c}$ a **co-rank tuple**.

By the definition of the action of G on $\mathrm{End}^{\times n}_{nil}(V)$ it is clear, that $g.f$ and f have the same co-rank tuple for any $g \in G$. Thus the co-rank tuples are constant on G-orbits in $\mathrm{End}^{\times n}_{nil}(V)$. It follows from Proposition 3.9 that all nilpotent endomorphisms with the same co-rank tuple are obtained in this way. Let f and f' be tuples in $\mathrm{End}^{\times n}_{nil}(V)$ which live in the same G-orbit. The element $g \in G$ such that $g.f = f'$ establishes an isomorphism of the fibres $\pi^{-1}(f)$ and $\pi^{-1}(f')$. Accordingly it is sufficient to study the fibre for one representative of the orbits.

The co-rank tuple of $f = (s_{-1}, \dots, s_{-1})$ in the approximation to the parameter N is denoted by $\mathbf{c}^0$ and its entries are $c^0_{i,k} = k+1$ since the co-rank of s_{-1} is one and the co-rank of $s_{-1} \circ s_{-1}$ is two. Accordingly the co-rank of the maps f_i is independent of the approximation and it is independent of the starting vertex i. For the endomorphism $f = \mathrm{pr} = (s_{-1} \circ \mathrm{pr}_1, \dots, s_{-1} \circ \mathrm{pr}_1)$ we denote its co-rank tuple by $\mathbf{c}^1$ and the entries of this tuple are given by $c^1_{i,k} = 2(k+1)$ because the co-rank of $s_{-1} \circ \mathrm{pr}_1$ is two. Again the co-rank is independent of the approximation parameter N.

Moreover this co-rank tuple satisfies the property

$$c_{i,k} = \sum_{l=0}^{k} c_{i+l,0} \tag{6.7.1}$$

which is also satisfied for $\mathbf{c}^0$. We define a partial order on the co-rank tuples, where

$$\mathbf{c}' \leq \mathbf{c} :\Longleftrightarrow c'_{i,k} \geq c_{i,k} \text{ for all } i \in \mathbb{Z}_n.$$

From now on we only want to consider co-rank tuples $\mathbf{c}$ with $\mathbf{c}^1 \leq \mathbf{c} \leq \mathbf{c}^0$ which also satisfy Property 6.7.1.

These tuples correspond to the G-orbits of the maps $f \in \mathrm{End}_{nil}^{\times n}(V)$ where each f_i is either the shifted projection $s_{-1} \circ \mathrm{pr}_1$ or the index shift s_{-1}. They are completely determined by their entries $c_{i,0}$ for $i \in \mathbb{Z}_n$. Hence it is sufficient to view co-rank tuples in $\mathbb{Z}^n$ if the full tuple for any approximation to the parameter N satisfies Property 6.7.1. Based on this observation want to simplify the notation for the co-rank tuples. For a co-rank tuple $\mathbf{c}$ as above we now take the tuple

$$(c_{i,0} - 1)_{i \in \mathbb{Z}_n}$$

and by abuse of notation also denote it by $\mathbf{c}$. The new co-rank tuple $\mathbf{c}^1 = (1, \dots, 1) \in \mathbb{Z}^n$ corresponds to the degeneration we studied in the previous sections of this chapter. And the new tuple $\mathbf{c}^0 = (0, \dots, 0) \in \mathbb{Z}^n$ is corresponding to the non-degenerate affine flag variety.

Let $\mathcal{F}l^{\mathbf{c}}\big(\widehat{\mathfrak{gl}}_n\big)$ be the linear degenerate affine flag variety corresponding to the co-rank tuple $\mathbf{c}$. The degenerate flag varieties for tuples with $\mathbf{c}^1 \leq \mathbf{c} \leq \mathbf{c}^0$ can be viewed as intermediate degenerations between the non-degenerate affine flag $\mathcal{F}l\big(\widehat{\mathfrak{gl}}_n\big)$ and the Feigin-degenerate affine flag variety $\mathcal{F}l^a\big(\widehat{\mathfrak{gl}}_n\big)$. This terminology is motivated by the structure of their approximations as given in the subsequent lemma.

Lemma 6.42. *For $\omega \in \mathbb{N}$ and $\mathbf{c} \in \{0,1\}^{\mathbb{Z}_n}$ the finite approximation is given as*

$$\mathcal{F}l^{\mathbf{c}}_\omega\big(\widehat{\mathfrak{gl}}_n\big) \cong \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}\big(M^{\mathbf{c}}_\omega\big),$$

where $\mathbf{e}_\omega := \mathbf{dim} \bigoplus_{i \in \mathbb{Z}_n} U_i(\omega n) = (\omega n)_{i \in \mathbb{Z}_n}$ and for every $i \in \mathbb{Z}_n$ the representation $M^{\mathbf{c}}_\omega$ contains the summand $U_i(\omega n) \otimes \mathbb{C}^2$ if $c_i = 1$ or $U_i(2\omega n)$ if $c_i = 0$.

Proposition 6.43. The quiver representation $M^{\mathbf{c}}_\omega$ is isomorphic to the quiver representation

$$\Big(M_{\alpha_i} := s_1 \circ \mathrm{pr}^{c_i}_{\omega n} \Big)_{i \in \mathbb{Z}_n}.$$

Proof. For the representation M_ω the vectorspace over each vertex $i \in \mathbb{Z}_n$ has dimension $2\omega n$. In the coefficient quiver of $M^{\mathbf{c}}_\omega$ there are $1 + c_i$ segments starting over the vertex $i \in \mathbb{Z}_n$.

The first segment is starting in the fist point over the vertex i and in the k-th step its arrow goes from the k-th point over the vertex $i + k - 1$ to the $k + 1$-th point over the vertex $i + k$. If $c_i = 1$ this segment has length ωn and there has to be a second segment starting over the same vertex. If $c_i = 0$ this segment has length $2\omega n$ and there is no second segment starting over the vertex $i \in \mathbb{Z}_n$.

Now assume that $c_i = 1$. The first segment ends in the ωn-th point over the vertex $i - 1$ and it is not possible that there exists an arrow pointing to the $\omega n + 1$-th point over the vertex i. We choose this point as starting point for the second segment starting over the vertex i.

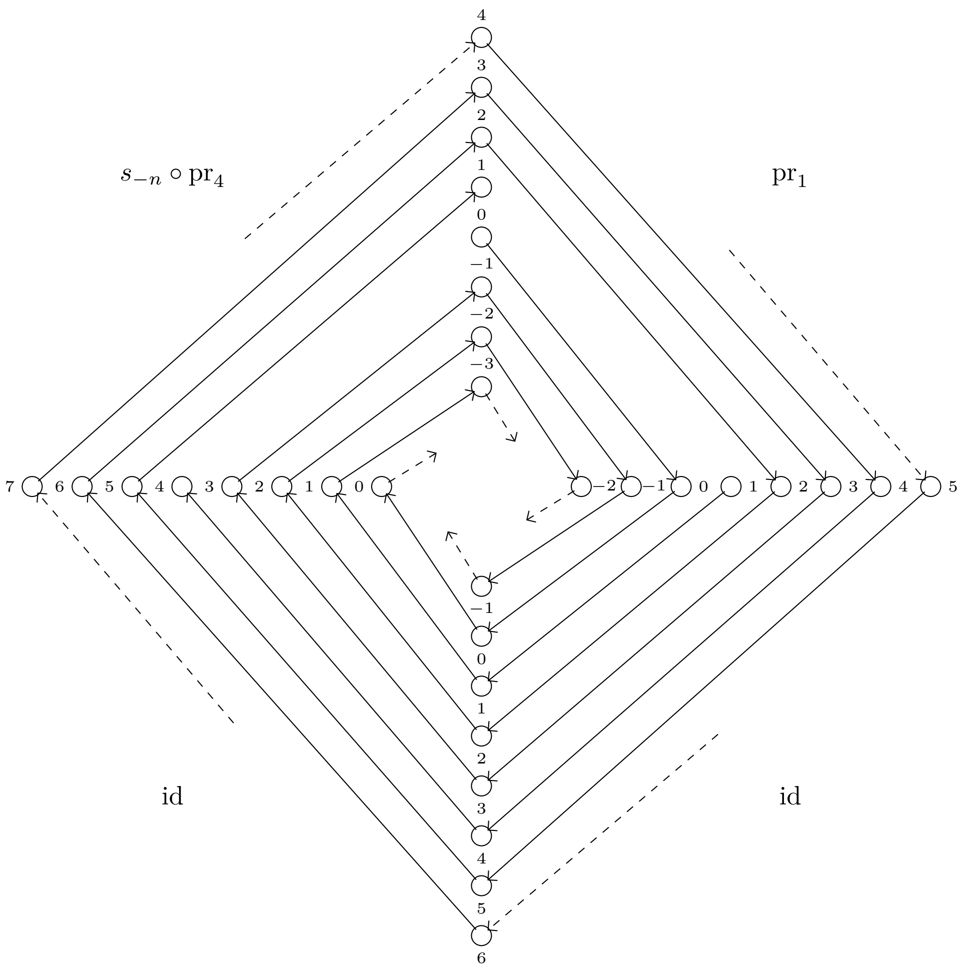
In the k-th step the arrow of this segment goes from the $\omega n + k$-th point over the vertex $i + k - 1$ to the $\omega n + k + 1$-th point over the vertex $i + k$ and it ends in the $2\omega n$-th point over the vertex $i + n - 1 = i - 1$. With this realisation of the coefficient quiver of $M_\omega^{\mathbf{c}}$ we have the maps

$$M_{\alpha_i} := s_1 \circ \mathrm{pr}_{\omega n}^{c_i}$$

for the arrow α_i from vertex i to vertex $i + 1$. □

We change the indices of the basis vectors over the vertices $i \in \mathbb{Z}_n$ in order to match the indices of the basis vectors for the spaces in the Sato Grassmannians SGr_i. This is done in the same way as for $X_\omega \oplus Y_\omega$ in the full degenerate setting since the vector spaces over the vertices of Δ_n have the dimension $2\omega n$ independent of the parameter $\mathbf{c}$.

EXAMPLE 6.44. Using the new labelling the coefficient quiver of $M_\omega^{\mathbf{c}}$ for $n = 4$, $\omega = 1$ and $\mathbf{c} = (1, 0, 0, 1)$ is given by



Again the dashed arrows in the picture indicate where the segments grow if we increase the value of ω.

Proof of Lemma 6.42. In this setting, it is possible to use the same maps as defined between the finite approximation of the $\mathbf{c}^1$-degenerate affine flag variety and the corresponding quiver Grassmannian in Theorem 6.4. For the basis of the vector spaces over the vertices of the quiver we take the same labelling and add arrows from $v_i^{(i)}$ to $v_i^{(i+1)}$ (resp. $v_{n-1}^{(n-1)}$ to $v_{-1}^{(0)}$) in the coefficient quiver if $c_i = 0$ (resp. $c_{n-1} = 0$). □

Remark. In the chapter about quiver Grassmannians for the equioriented cycle we discovered that it was crucial for certain properties of the quiver Grassmannian that the length of the projective and injective representations of Δ_n is a multiple of n. For the co-rank tuples $\mathbf{c}$ we restricted to, we still get approximations of the corresponding linear degenerate affine flag varieties $\mathcal{F}l^{\mathbf{c}}\big(\widehat{\mathfrak{gl}}_n\big)$ where the length of all summands of $M_\omega^{\mathbf{c}}$ is a multiple of n. This enables us to use the methods developed in Chapter 3 to study their approximations.

The approximation of the linear degenerate affine flag varieties $\mathcal{F}l^{\mathbf{c}}\big(\widehat{\mathfrak{gl}}_n\big)$ by quiver Grassmannians for the equioriented cycle would work for all co-rank tuples $\mathbf{c}$ coming from map tuples $f \in \mathrm{End}^{\times n}(V)$ where each f_i is an arbitrary finite composition of projections. But the resulting quiver Grassmanians can not be studied using the methods which are introduced in the previous chapters of this thesis.

6.8. Ind-Variety Structure

In this section we introduce closed embeddings between the approximations of the partial degenerations of the affine flag variety which are compatible with the ind-variety structure of the partial degenerate affine flag varieties.

6.8.1. Ind-Variety Structure of the Feigin Degeneration. Before we examine the maps for the ind-variety structure in the generality of the partial degenerations we construct them for the Feigin degeneration where the explicit description of the maps is less complicated. We realise the approximation for a given $\omega \in \mathbb{N}$ as a closed subset inside of the quiver Grassmannian for $\omega + 1$. For this purpose we need an other description of the cell which makes it easier to describe how the parametrisation of the cells changes along the embedding.

Proposition 6.45. The cells of the approximation $\mathcal{F}l_\omega^a\big(\widehat{\mathfrak{gl}}_n\big)$ are parametrised by the set

$$\mathcal{C}_\omega^a(n) := \Big\{ \big(I^{(i)}\big)_{i\in\mathbb{Z}_n} \in \prod_{i\in\mathbb{Z}_n} \binom{[2\omega n]}{\omega n} : s_2 I^{(i)} \cap [2\omega n] \subset I^{(i+1)} \text{ for all } i \in \mathbb{Z}_n \Big\}.$$

Proof. The map s_2 denotes the index shift by 2. Following Section 3.5.2 and applying the identification of the approximations with quiver Grassmannians we obtain that the cells are parametrised by the set

$$\Big\{ \mathbf{l} := (\ell_{i,1}, \ell_{i,2}) \in \bigoplus_{i\in\mathbb{Z}_n} [\omega n]_0 \times [\omega n]_0 : \ \mathbf{dim}\, U(\mathbf{l}) = \big(\omega n\big)_{i\in\mathbb{Z}_n} \Big\}.$$

Each number $\ell_{i,k}$ parametrises a segment of a successor closed subquiver.

From the structure of the coefficient quiver of $X_\omega \oplus Y_\omega$ we know that the indices of the vertices on a segment increase by 2 along the arrows of the quiver. The end points of the segments have the index $2\omega n$ if the segment corresponds to a summand of X_ω or $2\omega n - 1$ if the segment corresponds to a summand of Y_ω. With this information we can construct index sets corresponding to the labelling of the vertices on the segments from the length of the segments. This yields the claimed parametrisation of the cells. □

LEMMA 6.46. *For all $\omega \in \mathbb{N}$ there exists a closed embedding*

$$\Phi^a_\omega : \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega) \to \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_{\omega+1}}(X_{\omega+1} \oplus Y_{\omega+1})$$

preserving the dimension of the cells.

PROOF. The quiver Grassmannians $\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}(X_\omega \oplus Y_\omega)$ admit a cellular decomposition into attracting sets of torus fixed points. Following the proof of Theorem 4.10 the points in the cells are spanned by vectors

$$\Big\{ w^{(i)}_1, w^{(i)}_2, \dots, w^{(i)}_{\omega n} \Big\}$$

of the form

$$w^{(i)}_s = v^{(i)}_{k_s} + \sum_{j > k_s, j \notin I^{(i)}} \mu^{(i)}_{j,s} v^{(i)}_j$$

with coefficients $\mu^{(i)}_{j,s} \in \mathbb{C}$ for $i \in \mathbb{Z}_n$. Here $I^{(i)} \subset [2\omega n]$ for $i \in \mathbb{Z}_n$ are the index sets describing the corresponding torus fixed point as determined in Proposition 6.45.

Similarly as done for the approximations of the affine Grassmannian in Section 5.5, the points in the quiver Grassmannian which is isomorphic to the approximation of the affine flag variety can be described by a tuple of matrices

$$M^{(i)} \in M_{2\omega n, \omega n}(\mathbb{C})$$

for $i \in \mathbb{Z}_n$ collecting the coefficients $\mu^{(i)}_{j,s}$ of the basis vectors $v^{(i)}_j$ which parametrise the vectors $w^{(i)}_s$ spanning this point.

We define the map

$$\Psi^a_\omega : M_{2\omega n, \omega n}(\mathbb{C}) \to M_{2(\omega+1)n, (\omega+1)n}(\mathbb{C})$$

where

$$\Psi^a_\omega\big(M^{(i)}\big)_{p,q} := \begin{cases} m^{(i)}_{p-n,q} & \text{if } n < p \leq 2\omega n + n \text{ and } q \in [\omega n] \\ 1 & \text{if } q > \omega n \text{ and } p - 2\omega n = q - \omega n \\ 0 & \text{otherwise.} \end{cases}$$

These matrices have a block structure of the following shape

$$\Psi^a_\omega\big(M^{(i)}\big) = \begin{pmatrix} \mathbf{0}_{n,\omega n} & \mathbf{0}_{n,n} \\ M^{(i)} & \mathbf{0}_{\omega n, n} \\ \mathbf{0}_{n,\omega n} & \mathrm{id}_n \end{pmatrix}$$

where $\mathbf{0}_{p,q}$ is a $p \times q$ matrix with all entries equal to zero and id_n is the $n \times n$ identity matrix.

On the level of cells this corresponds to the map

$$\psi^a_\omega : \mathcal{C}^a_\omega(n) \to \mathcal{C}^a_{\omega+1}(n)$$

where

$$\psi^a_\omega\big(I^{(i)}\big) = s_n I^{(i)} \cup \Big\{ 2\omega n + n + 1, 2\omega n + n + 2, \dots, 2\omega n + 2n \Big\} \subseteq [2(\omega+1)n].$$

Hence the image of a fixed point is indeed a fixed point in the bigger approximation and the dimension is preserved since we do not add holes below the starting points of the segments.

The image of a point is a point in the attracting set of the image of the fixed point it is attracted from. This is checked in the same way as done for the points in the approximations of the degenerate affine Grassmannian □

COROLLARY 6.47. On the Feigin-degenerate affine flag variety $\mathcal{F}l^a\big(\widehat{\mathfrak{gl}}_n\big)$ the ind-topology and the Zariski topology coincide.

PROOF. It is clear that any point in the degenerate affine flag variety lives in some finite approximation. This approximation is isomorphic to a quiver Grassmannian. Combined with Lemma 6.46 the approximations by quiver Grassmannians induce an ind-variety structure on the affine flag variety. The topologies coincide since the cell structure is preserved by the embeddings [**71**, Proposition 7]. □

6.8.2. Ind-Variety Structure of the Non-Degenerate Affine Flag Variety. In this section we generalise the definition of the map between the approximations of the degenerate affine flag variety to the non-degenerate setting.

PROPOSITION 6.48. The cells of the approximation $\mathcal{F}l_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ are parametrised by the set

$$\mathcal{C}_\omega(n) := \Big\{ \big(I^{(i)}\big)_{i\in\mathbb{Z}_n} \in \prod_{i\in\mathbb{Z}_n} \binom{[2\omega n]}{\omega n} : s_1 I^{(i)} \cap [2\omega n] \subset I^{(i+1)} \text{ for all } i\in\mathbb{Z}_n \Big\}$$

PROOF. Analogous to the approximations of the degenerate affine flag variety we obtain that the cells are parametrised by the set

$$\Big\{ \mathbf{l} := (\ell_i) \in \bigoplus_{i\in\mathbb{Z}_n} [2\omega n]_0 : \ \mathbf{dim}\, U(\mathbf{l}) = \big(\omega n\big)_{i\in\mathbb{Z}_n} \Big\}$$

where each number ℓ_i parametrises a segment of a successor closed subquiver.

By the structure of the coefficient quiver of

$$M_\omega^0 := \bigoplus_{i\in\mathbb{Z}_n} U_i(2\omega n)$$

we know that the indices of the vertices on a segment increase by one along the arrows of the quiver. The end points of the segments have the index $2\omega n$. As in the degenerate case this information is sufficient to construct a bijection between the cells parametrised by numbers ℓ_i and the tuples of indices as defined above. □

LEMMA 6.49. *For all $\omega\in\mathbb{N}$ there exists a closed embedding*

$$\Phi_\omega : \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}\big(M^0_\omega\big) \to \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_{\omega+1}}\big(M^0_{\omega+1}\big)$$

preserving the dimension of the cells.

The proof of this statement is analogous to the proof of the generalisation of the map between the approximations of the degenerate affine Grassmannian to the map between the approximations of the non-degenerate affine Grassmannian.

COROLLARY 6.50. On the non-degenerate affine flag variety $\mathcal{F}l\big(\widehat{\mathfrak{gl}}_n\big)$ the ind-topology and the Zariski topology coincide.

6.8.3. Ind-Variety Structure of the Partial Degenerations. For the partial degenerations there are segments of two different lengths in the coefficient quiver of the representation $M_\omega^{\mathbf{c}}$. Thus it is not possible to parametrise the cells in a similar way as done for the approximations of the Feigin-degenerate and non-degenerate affine flag variety. The relation between the index sets $I^{(i)}$ and $I^{(i+1)}$ now depends on the value of c_i.

PROPOSITION 6.51. The cells of the approximation $\mathcal{F}l_\omega^{\mathbf{c}}(\widehat{\mathfrak{gl}}_n)$ are parametrised by the set

$$\mathcal{C}_\omega^{\mathbf{c}}(n) := \Big\{ \big(I^{(i)}\big)_{i\in\mathbb{Z}_n} \in \prod_{i\in\mathbb{Z}_n} \binom{[2\omega n]}{\omega n} : s_1 \mathrm{pr}_{\omega n}^{c_i} I^{(i)} \cap [2\omega n] \subset I^{(i+1)} \text{ for all } i \in \mathbb{Z}_n \Big\}$$

PROOF. This set parametrises the successor closed subquivers with ωn marked points over each vertex of the cycle in the coefficient quiver of the quiver representation

$$M_\omega^{\mathbf{c}} = \Big(M_{\alpha_i} := s_1 \circ \mathrm{pr}_{\omega n}^{c_i} \Big)_{i\in\mathbb{Z}_n}$$

which is isomorphic to the quiver representation providing the approximations of the partial degenerations. □

This choice of the index sets is not suitable to define the cell directly as the attracting set of the point which is spanned by the basis vectors with indices in the sets $I^{(i)}$. In order to obtain a cellular decomposition into attracting sets of torus fixed points we have to rearrange the segments such that there are no points below the end points of segments which are not the end point of some other segment.

This rearrangement corresponds to some permutations $\sigma^{(i)} \in S_{2\omega n}$. For the approximations of the non-degenerate affine flag the permutations are given by the identity whereas for the approximations of the Feigin degenerations we have to use the permutation

$$\sigma : [2\omega n] \to [2\omega n]$$
$$i \mapsto \begin{cases} 2i-1 & \text{if } i \le \omega n \\ 2(i-\omega n) & \text{otherwise.} \end{cases}$$

If we are in these special cases and apply this permutations to the set $\mathcal{C}_\omega^{\mathbf{c}}(n)$ we obtain the description of the cells by $\mathcal{C}_\omega^{a}(n)$ and $\mathcal{C}_\omega^{0}(n)$ as introduced before.

We define the index sets $J^{(i)} := \sigma^{(i)} I^{(i)}$ for

$$\big(I^{(i)}\big)_{i\in\mathbb{Z}_n} \in \mathcal{C}_\omega^{\mathbf{c}}(n)$$

which describe the torus fixed points

$$p = \Big(\mathrm{span}\big\{v_j : j \in J^{(i)}\big\}\Big)_{i\in\mathbb{Z}_n}.$$

This are the torus fixed points which allow us to define the cells in the approximation as their attracting sets.

Based on this description of the cells we can express the points in the approximations as done above explicitly for the Feigin degeneration. Using this parametrisation we can define maps between the approximations of the partial degenerations as done for the partial degenerations of the affine Grassmannian. Similar as in Chapter 5 we prove the subsequent properties of these maps.

LEMMA 6.52. *For all $\omega \in \mathbb{N}$ there exists a closed embedding*

$$\Phi^{\mathbf{c}}_{\omega} : \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}\big(M^{\mathbf{c}}_{\omega}\big) \to \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_{\omega+1}}\big(M^{\mathbf{c}}_{\omega+1}\big).$$

It preserves the dimension of the cells with all segments shorter than ωn.

6.9. Partial Degenerations of Affine Dellac Configurations

In this section we introduce subsets of affine Dellac configurations which describe the cells in the approximations of the partial degenerate affine flag varieties. For the approximations of the intermediate degenerations of the affine flag variety it is not possible to apply Theorem 4.10 in order to examine the cell structure because not all summands of the used quiver representation have the same length.

For every nilpotent representation $U \in \mathrm{rep}_{\mathbb{C}}(\Delta_n)$ there exists a parameter $N \in \mathbb{N}$ such that $U \in \mathrm{rep}_{\mathbb{C}}(\Delta_n, \mathrm{I}_N)$. Hence it is conjugated to a direct sum of indecomposable nilpotent representations, i.e.

$$U \cong U(\mathbf{d}) := \bigoplus_{i\in\mathbb{Z}_n} \bigoplus_{\ell=1}^{N} U(i;\ell) \otimes \mathbb{C}^{d_{i,\ell}}$$

where $d_{i,\ell} \in \mathbb{Z}_{\geq 0}$ for all $i \in \mathbb{Z}_n$ and $\ell \in [N]$. The number of indecomposable summands of $U(\mathbf{d})$ ending over the vertex $i \in \mathbb{Z}_n$ is given by

$$d_i := \sum_{\ell=1}^{N} d_{i,\ell}.$$

The condition $d(\alpha) := d_{t_\alpha}$ for all $\alpha \in \mathbb{Z}_n$ induces a grading of the vertices in the coefficient quiver of $U(\mathbf{d})$ which satisfies the assumptions of Theorem 4.7.

Moreover it is possible to prove the subsequent generalisation of Theorem 4.10 with the methods developed in Chapter 4.

THEOREM 6.53. *Let $M = U(\mathbf{d})$ and $\mathbf{e} \leq \mathbf{dim}\, M$. For every $L \in \mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(M)^T$, the subset $\mathcal{C}(L) \subseteq \mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(M)$ is an affine space and the quiver Grassmannian admits a cellular decomposition*

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(M) = \coprod_{L\in \mathrm{Gr}^{\Delta_n}_{\mathbf{e}}(M)^T} \mathcal{C}(L).$$

This result can be applied to the quiver Grassmannians approximating the partial degenerations of the affine flag variety. For every $i \in \mathbb{Z}_n$ the representation $M^{\mathbf{c}}_{\omega}$ contains the summand

$$U(i;\omega n) \otimes \mathbb{C}^2 \ \text{ if } c_i = 1 \qquad \text{or} \qquad U(i;2\omega n) \ \text{ if } c_i = 0.$$

Recall that for $\omega \in \mathbb{N}$ the finite approximation is given as

$$\mathcal{F}l^{\mathbf{c}}_{\omega}\big(\widehat{\mathfrak{gl}}_n\big) \cong \mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}\big(M^{\mathbf{c}}_{\omega}\big),$$

where $\mathbf{e}_\omega = (\omega n)_{i\in\mathbb{Z}_n}$.

By Proposition 4.9, the cells in the approximations are in bijection with successor closed subquivers in the coefficient quiver of $M^{\mathbf{c}}_{\omega}$ with ωn marked points over each vertex. Accordingly these successor closed subquivers are parametrised by the set

$$\mathcal{C}_{\mathbf{e},\mathbf{c}}\big(\Delta_n, \mathrm{I}_N\big) := \Big\{ \mathbf{l} := (\ell_{i,k}) \in \bigoplus_{i\in\mathbb{Z}_n} \bigoplus_{k=1}^{c_i+1} \big[(2-c_i)\omega n\big]_0 : \ \mathbf{dim}\, U(\mathbf{l}) = \mathbf{e} \Big\}$$

where

$$U(\mathbf{l}) = \bigoplus_{i\in\mathbb{Z}_n} \bigoplus_{k=1}^{c_i+1} U(i;\ell_{i,k}).$$

DEFINITION 6.54. An affine Dellac configuration $\widehat{D} \in \widehat{DC}_n(\omega)$ is called **c-degenerate** to the parameter $\mathbf{c} \in \{0,1\}^n$ if $k_j > 0$ for $j \in [n]$ implies that $k_{j+n} = \omega$ and $p_{j+n} = n$ whenever $c_j = 0$. The set of all **c**-degenerate affine Dellac configurations is denoted by $\widehat{DC}{}^{\mathbf{c}}_n(\omega)$.

These configurations parametrise the cells in the finite approximation $\mathcal{F}l^{\mathbf{c}}_\omega\big(\widehat{\mathfrak{gl}}_n\big)$.

THEOREM 6.55. *There is an isomorphism*

$$\widehat{DC}{}^{\mathbf{c}}_n(\omega) \cong \mathcal{C}_{\mathbf{e},\mathbf{c}}\big(\Delta_n, \mathrm{I}_N\big).$$

PROOF. For the special case with $c_j = 0$ for all $j \in [n]$ we obtain the Feigin degeneration. In this setting the isomorphism was established in Theorem 6.25.

In the general setting we distinguish two cases for the segments in the coefficient quiver of $M^{\mathbf{c}}_\omega$ coming from $c_j = 0$ or $c_j = 1$. For $c_j = 1$ there are two summands $U(j;\omega n)$ in $M^{\mathbf{c}}_\omega$. From the length of the subrepresentations $\ell_{j,1}$ and $\ell_{j,2}$ in these summands we can compute the entries k_j and k_{j+n} as well as their positions p_j and p_{j+n} in the configuration by

$$\ell_{j,1} = \min\{k_j - 1, 0\}\cdot n + p_j \quad \text{and} \quad \ell_{j,2} = \min\{k_{j+n} - 1, 0\}\cdot n + p_{j+n}.$$

For $c_j = 0$ there is only one summand $U(j;2\omega n)$ in $M^{\mathbf{c}}_\omega$. If $\ell_{j,1} \leq \omega n$ we set $k_j = 0$, $p_j = 0$ and compute k_{j+n} and p_{j+n} by

$$\ell_{j,1} = \min\{k_{j+n} - 1, 0\}\cdot n + p_{j+n}.$$

Otherwise we set $k_{j+n} = \omega$, $p_{j+n} = n$ and compute k_j and p_j by

$$\ell_{j,1} - \omega n = \min\{k_j - 1, 0\}\cdot n + p_j.$$

By the proof of the isomorphism

$$\widehat{DC}_n(\omega) \cong \Big\{\mathbf{l} := (\ell_{i,1}, (\ell_{i,2}) \in \bigoplus_{i\in\mathbb{Z}_n} [n]_0 \times [n]_0 : \ \mathbf{dim}\, U(\mathbf{l}) = \mathbf{e}\Big\}$$

it is clear that via this map a tuple $\mathbf{l} \in \mathcal{C}_{\mathbf{e},\mathbf{c}}(\Delta_n, \mathrm{I}_N)$ describes an affine Dellac configuration to the parameter ω. It follows from the construction of the map that this configuration also satisfies the assumption of **c**-degenerations.

Starting with a **c**-degenerate configuration the same assignments as above gives us a tuple $\mathbf{l}$ describing a cell in the set $\mathcal{C}_{\mathbf{e},\mathbf{c}}(\Delta_n, \mathrm{I}_N)$. □

This correspondence has some immediate consequences for the Euler Poincaré characteristic of the approximations $\mathcal{F}l^{\mathbf{c}}_\omega\big(\widehat{\mathfrak{gl}}_n\big)$. Let $\mathbf{c}^0$ be the tuple where every entry equals zero. This corresponds to the non-degenerate affine flag variety. The tuple $\mathbf{c}^1$ where every entry equals one describes the Feigin degeneration of the affine flag variety. On the tuples we define the partial order

$$\mathbf{c}' \leq \mathbf{c} :\Longleftrightarrow c'_i \geq c_i \text{ for all } i \in \mathbb{Z}_n.$$

COROLLARY 6.56. For two tuples $\mathbf{c}'$ and $\mathbf{c}$ with $\mathbf{c}^1 \leq \mathbf{c}' \leq \mathbf{c} \leq \mathbf{c}^0$, the Euler Poincaré characteristics of two approximations satisfy

$$\chi\,\mathcal{F}l^{\mathbf{c}}_\omega\big(\widehat{\mathfrak{gl}}_n\big) \ \leq\ \chi\,\mathcal{F}l^{\mathbf{c}'}_\omega\big(\widehat{\mathfrak{gl}}_n\big)$$

where equality holds if and only if the tuples are equal.

PROOF. The set $\widehat{DC}_n^{\mathbf{c}}(\omega)$ arises from $\widehat{DC}_n^{\mathbf{c}'}(\omega)$ by adding additional assumptions. If both tuples are equal there are no additional assumptions and both sets are equal. There exist configurations in the set $\widehat{DC}_n^{\mathbf{c}'}(\omega)$ which do not satisfy the additional assumptions. Hence the inequality is strict if and only if the inequality of the tuples is strict. □

This implies that for a parameter $\omega \in \mathbb{N}$ the Euler Poincaré characteristic of the approximation of the non-degenerate affine flag variety is strictly smaller than the Euler Poincaré characteristic of the approximation of the Feigin-degenerate affine flag variety. Moreover the examples in Appendix C.1 suggest that the characteristic only depends on the number of projections and not their positions.

CONJECTURE 6.57. *The Euler Poincaré characteristic of two approximations satisfies*

$$\chi\,\mathcal{F}l_\omega^{\mathbf{c}}\big(\widehat{\mathfrak{gl}}_n\big) \;\geq\; \chi\,\mathcal{F}l_\omega^{\mathbf{c}'}\big(\widehat{\mathfrak{gl}}_n\big)$$

if and only if

$$\sum_{i\in\mathbb{Z}_n} c_i \geq \sum_{i\in\mathbb{Z}_n} c_i'.$$

Equality holds if and only if both sums are equal.

The proof of this statement should somehow use the symmetries in the shape of the partial degenerate affine Dellac configurations. But the comparison of Euler characteristics is not one of the main goals of this thesis such that we do not want to go into further details here.

6.10. Poincaré Polynomials of the Approximations

In this section we develop a description of the Poincaré polynomials of the approximations $\mathcal{F}l_\omega^{\mathbf{c}}\big(\widehat{\mathfrak{gl}}_n\big)$ which is based on the partial degenerations of affine Dellac configurations. It utilises the identification of cells with successor closed subquivers and Dellac configuration. The formula to compute the dimension of a cell generalises the formula for the loop quiver which is defined in Section 5.1.5.

6.10.1. Poincaré Polynomials for the Approximations of the Feigin Degeneration. First we examine the case of the Feigin-degenerate affine flag variety. Let $\widehat{D}_{\mathbf{k}} \in \widehat{DC}_n(\omega)$ an affine Dellac configuration. Depending on $j \in \mathbb{Z}_{2n}$ and the position of k_j in the configuration $\widehat{D}_{\mathbf{k}}$ we define the index set

$$[j, j+p_j)_n := [j, j+p_j)_{\mathrm{mod}\ n} := \Big\{ i \in \mathbb{Z}_{2n} : 0 \leq (i-j) < p_j \bmod n \Big\}.$$

For $j \in \mathbb{Z}_n$ define the functions

$$\begin{aligned} h_j\big(\widehat{D}_{\mathbf{k}}\big) := {} & 2(p_j + nr_j) + \Big\lceil \frac{k_j}{\omega} \Big\rceil \Big\lfloor \frac{j}{n} \Big\rfloor \Big(\Big\lceil \frac{k_{j-n} + 1 - k_j}{\omega + 1} \Big\rceil - 1 \Big) \\ & - \sum_{i \in [j, j+p_j)_n} \min\{k_j, r_i\} - \sum_{i \in \mathbb{Z}_{2n} \setminus [j, j+p_j)_n} \min\{r_j, r_i\} \\ & - \Big| \Big\{ i \in [j, j+p_j)_n : 0 < k_i \leq k_j,\ p_i \geq p_j - (i-j) \bmod n \Big\} \Big| \\ & - \Big| \Big\{ i \in \mathbb{Z}_{2n} \setminus [j, j+p_j)_n : 0 < k_i \leq r_j,\ p_i \geq p_j + (j-i) \bmod n \Big\} \Big|. \end{aligned}$$

Using these functions on the affine Dellac configurations we can compute the dimension of the corresponding cells in the quiver Grassmannians.

PROPOSITION 6.58. Let $\widehat{D}_{\mathbf{k}} \in \widehat{DC}_n(\omega)$ an affine Dellac configuration. The dimension of the corresponding cell in the quiver Grassmannian

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}\Big(\bigoplus_{i\in\mathbb{Z}_n} U_i(\omega n) \otimes \mathbb{C}^2 \Big)$$

is given by

$$h\big(\widehat{D}_{\mathbf{k}}\big) := \sum_{j\in\mathbb{Z}_{2n}} h_j\big(\widehat{D}_{\mathbf{k}}\big).$$

PROOF. The dimension of a cell in this quiver Grassmannian equals the number of holes below the starting points of the segments in the successor closed subquiver corresponding to the cell. These segments are encoded in the k_j's and their positions in the configuration $\widehat{D}_{\mathbf{k}}$. It remains to show that for $j \in \mathbb{Z}_{2n}$ the function h_j counts the number of holes below the starting point of the segment corresponding to k_j.

The number

$$2(p_j + nr_j) - \Big\lceil \frac{k_j}{\omega} \Big\rceil \Big\lfloor \frac{j}{n} \Big\rfloor$$

is the height of the starting point of the segment which corresponds to k_j. Here $p_j + nr_j$ is the length of the segment and in each step we go up by two since there end two segments over each vertex $i \in \mathbb{Z}_{2n}$. The other term counts if we are in the upper or lower segment ending over the vertex j.

The index set $[j, j + p_j)_n$ contains the indices for segments which have a point below the starting point of the j-th segment and the distance of these points to the starting points of the corresponding segments is bigger than nr_j. Hence these segments have nk_j points below the starting point of the j-th segment.

All other segments have nr_j points below the starting point of the j-th segment and their indices are collected in the set $\mathbb{Z}_{2n} \setminus [j, j + p_j)_n$. Now we have to count the number of points in the subsegments which are described by k_i for $i \in \mathbb{Z}_{2n}$ and live below the starting point of the j-th segment.

For this purpose we distinguish the two cases introduced above. If $i \in [j, j + p_j)_n$ there are at least $\min\{k_j, r_i\}$ points of the i-th subsegment below the starting point of the j-th segment. For $0 < k_i \le k_j$ it depends on the position of k_i if the segment covers more than r_i holes. This is the case if

$$p_i \ge p_j - (i - j) \bmod n$$

and we have to remove one more hole. This case is captured by

$$\begin{aligned} &- \sum_{i\in[j,j+p_j)_n} \min\{k_j, r_i\} \\ &- \Big|\big\{ i \in [j, j + p_j)_n : 0 < k_i \le k_j,\ p_i \ge p_j - (i - j) \bmod n \big\}\Big|. \end{aligned}$$

The number

$$\Big\lceil \frac{k_j}{\omega} \Big\rceil \Big\lfloor \frac{j}{n} \Big\rfloor \Big\lceil \frac{k_{j-n} + 1 - k_j}{\omega + 1} \Big\rceil$$

corrects the number of holes if we are in the lower segment and the upper segment is longer than the lower segment. In this case we removed one hole to much by the above formula.

If $i \notin [j, j + p_j)_n$ there are at least $\min\{r_j, r_i\}$ points of the i-th subsegment below the starting point of the j-th segment. In this case the position of k_i is important if $0 < k_i \leq r_j$ and we have to remove an additional hole if

$$p_i \geq p_j + (j-i) \bmod n.$$

This is handled by the formula

$$\begin{aligned} &- \sum_{i \in \mathbb{Z}_{2n} \setminus [j, j+p_j)_n} \min\{r_j, r_i\} \\ &- \left| \left\{ i \in \mathbb{Z}_{2n} \setminus [j, j+p_j)_n : 0 < k_i \leq r_j,\ p_i \geq p_j + (j-i) \bmod n \right\} \right. . \end{aligned}$$

□

Remark. This generalises the formula for the affine Grassmannian as introduced in the previous chapter. Here the formula becomes more complicated because the numbers k_i and k_j are not sufficient to count the repetitions of the i-th segment below the j-th segment. For the affine Grassmannian these numbers were sufficient because the loop quiver has only one vertex. Nevertheless the structure of both formulas is the same. First we determine the height of the starting point of a segment. Then we count the repetitions of all segments which correspond to this cell and are below this starting point. The difference gives the number of holes below this segment. Summation over all segments determines the dimension of the corresponding cell.

Combining this formula with the result about the cellular decomposition of the approximation of the Feigin-degenerate affine flag variety we obtain the subsequent formula for their Poincaré polynomials.

Theorem 6.59. *For $\omega \in \mathbb{N}$, the Poincaré polynomial of $\mathcal{F}l^a_\omega(\widehat{\mathfrak{gl}}_n)$ is given by*

$$p_{\mathcal{F}l^a_\omega(\widehat{\mathfrak{gl}}_n)}(q) = \sum_{D \in \widehat{DC}_n(\omega)} q^{h(D)}.$$

6.10.2. Poincaré Polynomials for the Approximations of the Affine Flag Variety. Let $\widehat{D}_{\mathbf{k}} \in \widehat{DC}^{\mathfrak{c}^0}_n(\omega)$ an affine Dellac configuration which corresponds to a cell in the approximation of the non-degenerate affine flag variety. Define the index set

$$[j, j+p_j)_n^{\mathbb{Z}_n} := \Big\{ i \in \mathbb{Z}_n : 0 \leq (i-j) < p_j \bmod n \Big\}.$$

From the entries of the configuration we compute

$$\tilde{k_j} := k_j + k_{j+n}, \quad \tilde{r}_j := \max\{0, \tilde{k}_j - 1\} \quad \text{and} \quad \tilde{p}_j := \Big\lceil \frac{k_j}{\omega} \Big\rceil p_j + \Big(1 - \Big\lceil \frac{k_j}{\omega} \Big\rceil\Big) p_{j+n}$$

for $j \in \mathbb{Z}_n$. Here $\tilde{p}_j$ equals p_j if $k_j > 0$ and otherwise it is equal to p_{j+n}. Using these numbers we define the functions

$$\begin{aligned} h^0_j(\widehat{D}_{\mathbf{k}}) := {} & p_j + p_{j+n} + nr_j + nr_{j+n} \\ & - \sum_{i \in [j, j+p_j)_n^{\mathbb{Z}_n}} \min\{\tilde{k}_j, \tilde{r}_i\} - \sum_{i \in \mathbb{Z}_n \setminus [j, j+\tilde{p}_j)_n^{\mathbb{Z}_n}} \min\{\tilde{r}_j, \tilde{r}_i\} \\ & - \left| \left\{ i \in [j, j+\tilde{p}_j)_n^{\mathbb{Z}_n} : 0 < \tilde{k}_i \leq \tilde{k}_j,\ \tilde{p}_i \geq \tilde{p}_j - (i-j) \bmod n \right\} \right| \\ & - \left| \left\{ i \in \mathbb{Z}_n \setminus [j, j+\tilde{p}_j)_n^{\mathbb{Z}_n} : 0 < \tilde{k}_i \leq \tilde{r}_j,\ \tilde{p}_i \geq \tilde{p}_j + (j-i) \bmod n \right\} \right| \end{aligned}$$

for all $j \in \mathbb{Z}_n$ and

$$h^0(\widehat{D}_{\mathbf{k}}) := \sum_{j \in \mathbb{Z}_n} h^0_j(\widehat{D}_{\mathbf{k}}).$$

THEOREM 6.60. *For $\omega \in \mathbb{N}$, the Poincaré polynomial of $\mathcal{F}l_\omega(\widehat{\mathfrak{gl}}_n)$ is given by*

$$p_{\mathcal{F}l_\omega(\widehat{\mathfrak{gl}}_n)}(q) = \sum_{D \in \widehat{DC}^{\mathbf{c}^0}_n(\omega)} q^{h^0(D)}.$$

PROOF. In the coefficient quiver corresponding to the approximation $\mathcal{F}l_\omega(\widehat{\mathfrak{gl}}_n)$ there are n segments of length $2\omega n$ where exactly one segment ends over each vertex of the quiver. Hence the height of the starting point of the j-th segment equals the length of this segment. By the correspondence of cells and configurations the length of this segment is computed as

$$\tilde{p}_j + n\tilde{r}_j = p_j + p_{j+n} + nr_j + nr_{j+n}.$$

All relevant informations about the n segments are encoded in the numbers $\tilde{p}_j$ and $\tilde{k}_j$ for $j \in \mathbb{Z}_n$ such that the repetitions of the segments below the starting point of the j-th segment can be computed from these numbers as in the setting of the Feigin degeneration. The only difference is that there is only one segment for each $j \in \mathbb{Z}_n$ such that we can remove the other cases from the formula for the Feigin degeneration. □

6.10.3. Poincaré Polynomials for the Approximations of the Partial Degenerations. Let $\mathbf{c}$ be a corank tuple and $\widehat{D}_\mathbf{k} \in \widehat{DC}^{\mathbf{c}}_n(\omega)$ a partial degenerate affine Dellac configuration. Depending on the tuple $\mathbf{c}$ we define the index set

$$I_\mathbf{c} := \Big\{ i \in \mathbb{Z}_{2n} : i < n \text{ or } i \geq n \text{ and } c_{i-n+1} = 1 \Big\}.$$

From the entries of the configuration we compute the numbers

$$\tilde{k}_j := \begin{cases} k_j + k_{j+n} & \text{if } c_{j+1} = 0 \\ k_j & \text{otherwise} \end{cases}, \quad \tilde{p}_j := \begin{cases} p_{j+n} & \text{if } c_{j+1} = 0 \text{ and } p_j = 0 \\ p_j & \text{otherwise.} \end{cases}$$

and

$$\tilde{r}_j := \max\big\{0, \tilde{k}_j - 1\big\}, \quad s_j := \min\big\{\omega n, \tilde{p}_j + n\tilde{r}_j\big\}, \quad t_j := \max\big\{0, \tilde{p}_j + n\tilde{r}_j - \omega n\big\}.$$

For $j \in I_\mathbf{c}$ we use the index set

$$[j, j+p_j)_n^{I_\mathbf{c}} := [j, j+p_j)_n \cap I_\mathbf{c}$$

to define the functions

$$\begin{aligned} h_j^\mathbf{c}(\widehat{D}_\mathbf{k}) := \; & \tilde{p}_j + n\tilde{r}_j + \sum_{\ell=t_j+1}^{s_j} c_{j+\ell-1} + \Big\lceil \frac{k_j}{\omega} \Big\rceil \Big\lfloor \frac{j}{n} \Big\rfloor \Big(\Big\lceil \frac{k_{j-n}+1-k_j}{\omega+1} \Big\rceil - 1 \Big) \\ & - \sum_{i \in [j,j+p_j)_n^{I_\mathbf{c}}} \min\{\tilde{k}_j, \tilde{r}_i\} - \sum_{i \in I_\mathbf{c} \setminus [j,j+\tilde{p}_j)_n^{I_\mathbf{c}}} \min\{\tilde{r}_j, \tilde{r}_i\} \\ & - \Big| \Big\{ i \in [j, j+\tilde{p}_j)_n^{I_\mathbf{c}} : 0 < \tilde{k}_i \leq \tilde{k}_j,\ \tilde{p}_i \geq \tilde{p}_j - (i-j) \bmod n \Big\} \Big| \\ & - \Big| \Big\{ i \in I_\mathbf{c} \setminus [j, j+\tilde{p}_j)_n^{I_\mathbf{c}} : 0 < \tilde{k}_i \leq \tilde{r}_j,\ \tilde{p}_i \geq \tilde{p}_j + (j-i) \bmod n \Big\} \Big| \end{aligned}$$

and their sum

$$h^\mathbf{c}(\widehat{D}_\mathbf{k}) := \sum_{j \in \mathbb{Z}_n} h_j^\mathbf{c}(\widehat{D}_\mathbf{k}).$$

THEOREM 6.61. *For $\omega \in \mathbb{N}$ and $\mathbf{c} \in \mathbb{Z}^n$ with $\mathbf{c}^1 \leq \mathbf{c} \leq \mathbf{c}^0$, the Poincaré polynomial of $\mathcal{F}l^{\mathbf{c}}_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ is given by*

$$p_{\mathcal{F}l^{\mathbf{c}}_\omega\left(\widehat{\mathfrak{gl}}_n\right)}(q) = \sum_{D \in \widehat{DC}^{\,\mathbf{c}}_n(\omega)} q^{h^{\mathbf{c}}(D)}.$$

PROOF. The subsegments in the coefficient quiver for the approximation

$$\mathcal{F}l^{\mathbf{c}}_\omega\big(\widehat{\mathfrak{gl}}_n\big)$$

which parametrise the cell of the approximation can be indexed by the set $I_{\mathbf{c}}$ and they are determined by the numbers $\tilde{k}_j$ and $\tilde{p}_j$ for $j \in I_{\mathbf{c}}$.

For the first ωn steps in each step a segment moves up by the number of segments ending in the target vertex of the step. If i is the index of this vertex, the number of segments is given by $1 + c_i$. For the steps beyond ωn the segment moves up if and only if a segment of length $2\omega n$ ends in the vertex. The number of these segments ending over the vertex i is given by $1 - c_i$. Accordingly the height of the starting point of the j-th segment is computed as

$$\begin{aligned}\sum_{\ell=1}^{s_j}\big(1 + c_{j+\ell-1}\big) + \sum_{\ell=1}^{t_j}\big(1 - c_{j+\ell-1}\big) &= s_j + t_j + \sum_{\ell=1}^{s_j} c_{j+\ell-1} - \sum_{\ell=1}^{t_j} c_{j+\ell-1} \\ &= \tilde{p}_j + n\tilde{r}_j + \sum_{\ell=t_j+1}^{s_j} c_{j+\ell-1}\end{aligned}$$

where we have to subtract

$$\left\lceil \frac{k_j}{\omega} \right\rceil \left\lfloor \frac{j}{n} \right\rfloor$$

to distinguish between the lower and upper segment ending over the vertex j if both of them exist.

Here it is important that $t_j > 0$ implies that $s_j = \omega n$ and the maximal value for t_j is ωn because the maximal length for the segments is $2\omega n$. The computation of the repetitions of the other segments below the starting point of the j-th segment is analogous to the computation for the Feigin degeneration and non-degenerate affine flag variety. □

In examples we can compute the Poincaré polynomials based on this parametrisation of the cells by configurations and with the dimension function defined on the configurations. Alternatively we can draw the successor closed subquivers based on the parametrisation by the length of the subsegments and count the holes below the starting points of the segments. Based on the second approach we computed the Poincaré polynomials of some approximations using SageMath [**72**]. The results of these computations are presented in Appendix C.1 and in Appendix B.1 we provide the code of the program.

6.11. The Action of the Automorphism Groups in the Limit

In this section we examine the structure of the automorphism group $\mathrm{Aut}_{\Delta_n}(M^{\mathbf{c}}_\omega)$ and its action on the quiver Grassmannian

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}\big(M^{\mathbf{c}}_\omega\big)$$

for the case $\mathbf{c} \in \{\mathbf{c}^0, \mathbf{c}^1\}$. Based on this information we construct an embedding $\varphi^{\mathbf{c}}_\omega$ of the automorphism groups which is compatible with the map $\Phi^{\mathbf{c}}_\omega$.

For this construction we have to compute the elements of

$$\mathrm{Hom}_{\Delta_n}\big(U_i(\omega n), U_j(\omega n)\big)$$

for $i, j \in \mathbb{Z}_n$. The representation $U_i(\omega n)$ corresponds to the tuple $(M_\alpha)_{\alpha\in\mathbb{Z}_n}$ of maps

$$M_\alpha : \mathbb{C}^\omega \to \mathbb{C}^\omega$$

where $M_\alpha = \mathrm{id}_\omega$ if $t_\alpha \neq i$ and $M_\alpha = s_1$ if $t_\alpha = i$. In the same way let $(N_\alpha)_{\alpha\in\mathbb{Z}_n}$ be the tuple of maps corresponding to the indecomposable representation $U_j(\omega n)$.

A morphism from $U_i(\omega n)$ to $U_j(\omega n)$ is a tuple of maps

$$\varphi_k : \mathbb{C}^\omega \to \mathbb{C}^\omega$$

for $k \in \mathbb{Z}_n$ such that $\varphi_{t_\alpha} \circ M_\alpha = N_\alpha \circ \varphi_{s_\alpha}$ for all $\alpha \in \mathbb{Z}_n$. If $t_\alpha \notin \{i, j\}$ this yields $\varphi_{t_\alpha} = \varphi_{s_\alpha}$ which is equivalent to $\varphi_k = \varphi_{k-1}$ for $k \notin \{i, j\}$. Accordingly these maps satisfy $\varphi_k = \varphi_i$ for $k \in \{i, i+1, \dots, j-1\} \subset \mathbb{Z}_n$. For $t_\alpha = i$ we obtain $\varphi_{i-1} = \varphi_i \circ s_1$ such that $\varphi_k = \varphi_i \circ s_1$ holds for $k \in \{j, j+1, \dots, i-1\} \subset \mathbb{Z}_n$. The final relation is

$$\varphi_j = \varphi_i \circ s_1 = s_1 \circ \varphi_i = s_1 \circ \varphi_{j-1}$$

which is obtained for $t_\alpha = j$. Hence we can compute all maps φ_k for $k \in \mathbb{Z}_n$ after we have found a map φ_i commuting with the index shift s_1. This is the same commutativity relation as for endomorphisms of the representation A_N of the loop quiver with $\omega = N$. Following the same computation as in Section 5.6 of the chapter about the affine Grassmannian we obtain the subsequent parametrisation of the homomorphisms.

For a tuple $(\lambda_k)_{k\in[\omega]}$ with entries $\lambda_k \in \mathbb{C}$ we define the entries of the lower triangular matrix $A(\lambda) \in M_\omega(\mathbb{C})$ by

$$a_{i,j} := \begin{cases} \lambda_k & \text{if } k = i - j + 1 \\ 0 & \text{otherwise.} \end{cases}$$

This matrix describes an element of

$$\mathrm{Hom}_{\Delta_n}\big(U_i(\omega n), U_j(\omega n)\big)$$

for $i, j \in \mathbb{Z}_n$ by defining φ_i as the left multiplication with this matrix. All homomorphisms between $U_i(\omega n)$ and $U_j(\omega n)$ have a unique description of this form.

In the same way as in Section 5.6 we obtain the parametrisation of the elements of the automorphism group $\mathrm{Aut}_{\Delta_n}(M_\omega)$ where

$$M_\omega = \bigoplus_{i\in\mathbb{Z}_n} U_i(2\omega n).$$

For a tuple

$$\lambda := \big(\lambda_k^{(i,j)}\big) \text{ with } k \in [2\omega] \text{ and } i, j \in [n]$$

let $M_k(\lambda) \in M_n(\mathbb{C})$ be the matrix with entries $\lambda_k^{(i,j)}$ for $i, j \in [n]$. Define the $2\omega \times 2\omega$ block matrix $A_\lambda \in M_{2\omega n}(\mathbb{C})$ with the blocks

$$A_{p,q} := \begin{cases} M_k(\lambda) & \text{if } k = p - q + 1 \\ \mathbf{0}_{n,n} & \text{otherwise.} \end{cases}$$

Independent of the choice of λ this describes an endomorphism of M_ω where the maps φ_i over the vertices $i \in \mathbb{Z}_n$ are obtained form the matrix A_λ as described above for the homomorphisms between two indecomposable representations of the same length. Moreover all endomorphisms of M_ω admit a parametrisation of this form. If we additionally require that the matrix $M_1(\lambda) \in M_n(\mathbb{C})$ is invertible it is sufficient to describe the automorphisms of M_ω. Hence the group $\mathrm{Aut}_{\Delta_n}(M_\omega)$ is $2\omega n^2$-dimensional.

With this parametrisation for the elements of the automorphism group of M_ω we define the embedding

$$\varphi_\omega : \mathrm{Aut}_{\Delta_n}(M_\omega) \to \mathrm{Aut}_{\Delta_n}(M_{\omega+1})$$

by $\varphi_\omega(A_\lambda) := A_{\hat\lambda} \in M_{2(\omega+1)n}(\mathbb{C})$ where $\hat\lambda$ is obtained from λ as

$$\hat\lambda_k^{(i,j)} := \begin{cases} \lambda_k^{(i,j)} & \text{if } k < 2\omega \\ 0 & \text{otherwise.} \end{cases}$$

This embedding is compatible with the action of the automorphism group on the approximations of the affine flag variety.

LEMMA 6.62. *Let $A \in \mathrm{Aut}_{\Delta_n}(M_\omega)$ be an automorphism of the quiver Grassmannian*

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}\big(M_\omega\big) \cong \mathcal{F}l_\omega\big(\widehat{\mathfrak{gl}}_n\big).$$

Then the diagram

$$\begin{array}{ccc} \mathcal{F}l_\omega\big(\widehat{\mathfrak{gl}}_n\big) & \xrightarrow{\quad A \quad} & \mathcal{F}l_\omega\big(\widehat{\mathfrak{gl}}_n\big) \\ \Phi_\omega \big\downarrow & /\!\!/\!\!/ & \big\downarrow \Phi_\omega \\ \mathcal{F}l_{\omega+1}\big(\widehat{\mathfrak{gl}}_n\big) & \xrightarrow[\varphi_\omega(A)]{} & \mathcal{F}l_{\omega+1}\big(\widehat{\mathfrak{gl}}_n\big) \end{array}$$

commutes.

PROOF. The prove is based on the same arguments as for the approximations of the affine Grassmannian. The points in the quiver Grassmannians providing the approximations are now described as tuples of vector spaces which are spanned by certain vectors $w_k^{(i)}$ for $i \in \mathbb{Z}_n$ and $k \in [\omega n]$. The automorphisms act via left multiplication on these vectors. Now we have to compute for each vertex $i \in \mathbb{Z}_n$ that the action on the span is equivariant for the embedding of the span in the bigger approximation. Each of these computations is analogous to the computation for the action on the approximations of the affine Grassmannian. □

The elements of the automorphism group $\mathrm{Aut}_{\Delta_n}(M^a_\omega)$ where

$$M^a_\omega = \bigoplus_{i \in \mathbb{Z}_n} U_i(\omega n) \otimes \mathbb{C}^2$$

are parametrised by tuples

$$\mu := \big(\mu_k^{(i,j)}\big) \text{ with } k \in [\omega] \text{ and } i,j \in [2n]$$

such that the determinant of the matrix $M_1(\mu) \in M_{2n}(\mathbb{C})$ is non-zero. This is obtained as combination of the arguments in Section 5.6 and the parametrisation for the automorphisms of M_ω. Hence the dimension of the group $\mathrm{Aut}_{\Delta_n}(M^a_\omega)$ is computed as $4\omega n^2$.

We define the embedding

$$\varphi^a_\omega : \mathrm{Aut}_{\Delta_n}(M^a_\omega) \to \mathrm{Aut}_{\Delta_n}(M^a_{\omega+1})$$

by $\varphi^a_\omega(A_\mu) := A_{\hat{\mu}} \in M_{2(\omega+1)n}(\mathbb{C})$ where $\hat{\mu}$ is obtained from μ as

$$\hat{\mu}_k^{(i,j)} := \begin{cases} \mu_k^{(i,j)} & \text{if } k < \omega \\ 0 & \text{otherwise.} \end{cases}$$

Remark. This embedding is not compatible with the action of the automorphisms on the approximations of the degenerate affine flag variety and there exists no embedding compatible with this action. This is shown with the same arguments as for the degeneration of the affine Grassmannian.

Nevertheless we can prove the subsequent equivariance of the action with two consecutive embeddings of approximations using the methods developed in Section 5.6 and apply it to the approximations of the degenerate affine flag variety as described above for the non-degenerate setting.

Lemma 6.63. *Let $A \in \mathrm{Aut}_{\Delta_n}(M^a_\omega)$ be an automorphism of the quiver Grassmannian*

$$\mathrm{Gr}^{\Delta_n}_{\mathbf{e}_\omega}\big(M^a_\omega\big) \cong \mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big).$$

Then the diagram

$$\begin{CD} \mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big) @>{A}>> \mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big) \\ @V{\Phi^a_{\omega+1}\circ\Phi^a_\omega}VV /\!/\!/ @VV{\Phi^a_{\omega+1}\circ\Phi^a_\omega}V \\ \mathcal{F}l^a_{\omega+2}\big(\widehat{\mathfrak{gl}}_n\big) @>>{\varphi^a_{\omega+1}\circ\varphi^a_\omega(A)}> \mathcal{F}l^a_{\omega+2}\big(\widehat{\mathfrak{gl}}_n\big) \end{CD}$$

commutes.

CHAPTER 7

Equivariant Cohomology and the Moment Graph

The GKM or moment graph associated to an algebraic variety captures information about the structure of fixed points and one-dimensional orbits of the action of an algebraic torus on the variety. It is useful to describe geometric properties of the variety as for example cohomology and intersection cohomology. The foudation of this area was given by T. Braden, M. Goresky, R. Kottwitz and R. MacPherson in [**39, 11**]. In this chapter we follow the computation of the moment graph for degenerate flag varieties by G. Cerulli Irelli, E. Feigin and M. Reineke in [**21**]. For introductionary surveys about GKM theory and equivariant cohomology see for example [**54, 73**].

Let X be a projective algebraic variety over $\mathbb{C}$ with an algebraic action of a torus $T \cong (\mathbb{C}^*)^d$ which has finitely many fixed points and one-dimensional orbits. Moreover we assume that there exists an embedding $\mathbb{C}^* \hookrightarrow T$ such that the $\mathbb{C}^*$-fixed points and the T-fixed points coincide and the $\mathbb{C}^*$-action induces a cellular decomposition of X into T-invariant attracting sets of $\mathbb{C}^*$-fixed points. Classically it is assumed that X admits a T-invariant Whitney stratification by affine spaces [**11**, Section 1.1]. But for our application it is necessary to require the existence of the cellular decomposition. At some point both notions have to be related in the setting of the quiver Grassmannians which we want to consider.

Definition 7.1. The vertex set of the **moment graph** $\Gamma := \Gamma(X,T)$ is given by the set of torus fixed points, i.e.

$$\Gamma(X,T)_0 := X^T.$$

Let $\mathcal{O}^1_T(X)$ be the set of one-dimensional T-orbits in X. Each orbit $L \in \mathcal{O}^1_T(X)$ has two distinct limit points. On one hand the T-fixed point p of the cell C_p which contains the orbit L and on the other hand one other T-fixed point q in some cell C_q in the closure of C_p. The edges of Γ are given by the one-dimensional T-orbits, i.e.

$$\Gamma(X,T)_1 := \mathcal{O}^1_T(X)$$

and they are oriented as $s_L := q, t_L := p$ if $C_q \subset \overline{C_p}$.

Moreover we define a labelling of the edges as follows: Let $\mathfrak{t}$ be the Lie algebra of the torus T. All points in the one-dimensional T-orbit L have the same stabiliser in T, its Lie algebra is a hyperplane in $\mathfrak{t}$ and the annihilator of the hyperplane is denoted by α_L which will be the label of the edge L.

On the vertex set of the moment graph we define the partial order $q \leq p$ if and only if $C_q \subset \overline{C_p}$. This partial order and the induced Alexandrov topology is important for the computation of the Braden-MacPherson sheaves (BMP-sheaves) over moment graphs [**55**, Definition 3.6]. These sheaves can be used to compute the T-equivariant intersection cohomology of X [**11**, Theorem 1.5]. In the rest of this chapter we introduce a combinatoric approach to construct the moment graph and the labelling of its edges for the quiver Grassmannians which admit a cellular decomposition into attracting sets of $\mathbb{C}^*$-fixed points.

7.1. The Euler-Poincaré Graph of a Quiver Grassmannian

Let M be a representation of a quiver Q such that the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{Q}(M)$ has property (C). This means that there is a $\mathbb{C}^*$-action on $\mathrm{Gr}_{\mathbf{e}}^{Q}(M)$ inducing a cellular decomposition into attracting sets of $\mathbb{C}^*$-fixed points. These cells have a combinatoric description by successor closed subquivers of dimension type $\mathbf{e}$ in the coefficient quiver of M, i.e.

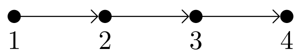
$$\mathrm{SC}_{\mathbf{e}}^{Q}(M) := \Big\{ T \overset{\rightarrow}{\subset} \Gamma(M, \mathcal{B}_\bullet) : |T_0 \cap \mathcal{B}_i| = e_i, \text{ for all } i \in Q_0 \Big\}.$$

If we order the segments of M according to the power with which $\lambda \in \mathbb{C}^*$ acts on the corresponding basis vectors, the dimension of a cell equals the number of holes below the starting points of the segments of T parametrising the cell.

For $S, T \in \mathrm{SC}_{\mathbf{e}}^{Q}(M)$ we say that T is obtained from S by a **fundamental mutation** and write $\mu(S) = T$ if we obtain the coefficient quiver T from S by moving up exactly one part of a segment where the order of segments is induced by the $\mathbb{C}^*$-action. The inverse fundamental mutations are obtained as downwards movements of subsegments.

Remark. The distance of the movement in a mutation is not limited such that it can happen that a fundamental mutation is the concatenation of two other fundamental mutations.

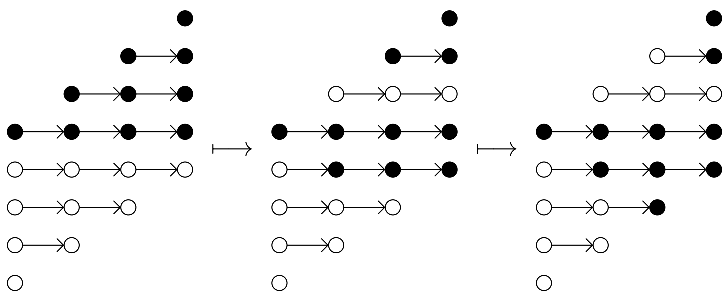
Example 7.2. Let Q be an equioriented quiver of type A_4, i.e.



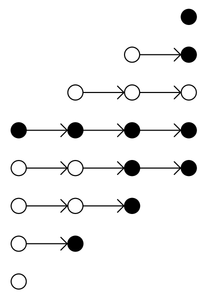
and let $M = X \oplus Y$ be a representation of Q where

$$X = \bigoplus_{i=1}^{4} P_i \quad \text{and} \quad Y = \bigoplus_{j=1}^{4} I_j$$

and $\mathbf{e} := \mathbf{dim}\, X$. The coefficient quiver for the top-dimensional cell of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{Q}(M)$ is given on the left below and after two inverse fundamental mutations we arrive at the right picture



From this coefficient quiver we can obtain the subsequent quiver via one or two inverse fundamental mutations



Definition 7.3. The vertices of the **Euler-Poincaré graph** $\mathrm{EP}_{\mathbf{e}}^{Q}(M)$ for the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{Q}(M)$ are given by the $\mathbb{C}^*$-fixed points, i.e.

$$\mathrm{EP}_{\mathbf{e}}^{Q}(M)_0 := \mathrm{SC}_{\mathbf{e}}^{Q}(M).$$

In the i-th row of the graph write the vertices corresponding to cells of dimension k where $i := \dim \mathrm{Gr}_{\mathbf{e}}^{Q}(M) - k + 1$ and label this row by k. For $p, q \in \mathrm{EP}_{\mathbf{e}}^{Q}(M)_0$ draw an arrow $\alpha \in \mathrm{EP}_{\mathbf{e}}^{Q}(M)_1$ with $s_\alpha = p$ and $t_\alpha = q$ if there exists a fundamental mutation of coefficient quivers such that $\mu(p) = q$.

The fundamental mutations increase the dimension of the cells such that in this graph all edges are oriented from the bottom to the top. For a certain class of quiver representations this graph captures the information about the inclusion relations of the cells in the decomposition of the corresponding quiver Grassmannian. The representation from the example above belongs to this class.

Based on this combinatoric data we want to describe the moment graph associated to the corresponding quiver Grassmannian as described for the degenerate flag variety by G. Cerulli Irelli, E. Feigin and M. Reineke in [**21**] and for complex algebraic varieties by T. Braden and R. MacPherson in [**11**]. For this construction it is necessary that the EP-graph associated to a quiver Grassmannian with a cellular decomposition satisfies the subsequent condition. If T is obtained from S by a fundamental mutation the corresponding cells satisfy $C(S) \subset \overline{C(T)}$.

This means that with a suitable label function this graph has the structure of a moment graph on a lattice as defined by P. Fiebig in [**31**]. Here the order of the vertices is induced by the order of the orbits as defined earlier in this chapter. Having this interpretation of the EP-graph it would remain to find a bijection between one-dimensional T-orbits and fundamental mutations. Then we can choose as label function for the EP-graph the labels which arise from the T-action. It is not clear yet in which generality these assumptions can be satisfied for the EP-graphs associated to quiver Grassmannians.

If we restrict the arrows to basic mutations, i.e. fundamental mutations of minimal distance, the Euler-Poincaré graph would have the structure of the Hasse-diagram for the cells ordered by inclusion with the additional information about the dimension of the cells.

The number of vertices in the Euler-Poincaré graph equals the Euler characteristic of the quiver Grassmannian, i.e.

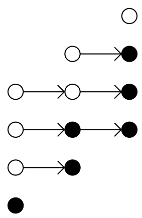
$$\chi\Big(\mathrm{Gr}_{\mathbf{e}}^{Q}(M)\Big) = |\mathrm{EP}_{\mathbf{e}}^{Q}(M)_0|$$

and the coefficient of q^k in the Poincaré polynomial of the quiver Grassmannian is computed as the number of vertices in the row of $\mathrm{EP}_{\mathbf{e}}^{Q}(M)$ indexed by k.

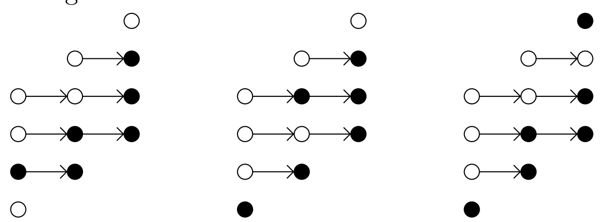
REMARK. In praxis the Euler-Poincaré graph can be computed by applying fundamental mutations to the zero-dimensional cell of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{Q}(M)$.

The idea of this chapter is to define a T-action on these quiver Grassmannians such that the corresponding moment graph and its label can be computed based on the combinatorics of the coefficient quivers.

For the quiver Grassmannian with $n = 3$ from the example above the zero-dimensional cell is parametrised by



Applying basic mutations we can for example compute the following coefficient quivers corresponding to one-dimensional cells.



In this setting the effective pairs labelling the edges of the moment graph as in [**21**] are given as the index of the segment where the fundamental mutation ends and the index of the segment where the movement of the subsegment starts. Here we orient the edges in the opposite direction as in [**21**].

7.2. Torus Actions on Quiver Grassmannians for the Loop

In this section we introduce the action of a torus T on quiver Grassmannians for the loop quiver. It is defined to be compatible with the cellular decomposition which is induced by the $\mathbb{C}^*$-action on these quiver Grassmannians. This means that the $\mathbb{C}^*$-fixed points and the T-fixed points coincide and the cells are stable under the action of T. For the special case of the quiver Grassmannians providing finite dimensional approximations of the affine Grassmannian this is linked to the construction arising from the combinatorics of the words in the corresponding affine Weyl group as studied by M. Lanini in [**56**].

As in the chapter about the affine Grassmannian we want to consider quiver Grassmannians of the form

$$Gr_{xN}\big(A_N \otimes \mathbb{C}^{(x+y)N}\big).$$

Recall that the quiver representation $A_N \otimes \mathbb{C}^m$ is isomorphic to the representation

$$M_{m,N} := \Big(\big(\mathbb{C}^{mN}\big), \big(s_m\big)\Big).$$

The vector space $\mathbb{C}^{mN}$ has a basis $v_{i,\ell}$ for $i \in [m]$ and $\ell \in [N]$. For

$$v = \sum_{i\in[m]} \sum_{\ell\in[N]} \mu_{i,\ell} v_{i,\ell}$$

define $s_i := \min\{\ell \in [N] : \mu_{i,\ell} \neq 0\}$ for all $i \in [m]$. We define the torus action

$$T := \big(\mathbb{C}^*\big)^{mN} \times \mathbb{C}^{mN} \longrightarrow \mathbb{C}^{mN}$$
$$(\lambda, v) \longmapsto \sum_{i\in[m]} \lambda_{i+m(s_i-1)} \sum_{\ell=s_i}^{N} \mu_{i,\ell} v_{i,\ell}.$$

In this way T acts with the same λ_j on all basis vectors corresponding to one of the m subsegements describing a $\mathbb{C}^*$-fixed point in the quiver Grassmannian and on each of the segments it acts with a different λ_j. Accordingly the $\mathbb{C}^*$-fixed points and the cells of the quiver Grassmannians above are stable under this action.

We define the embedding $T_0 := \big(\mathbb{C}^*\big)^m \times \mathbb{C}^* \hookrightarrow T$ via

$$\lambda_{i+m(s_i-1)} := \gamma_i \cdot \gamma_0^{s_i}$$

for $\big((\gamma_i)_{i\in[m]}, \gamma_0\big) \in T_0$.

It follows from the explicit parametrisation of the cells in the quiver Grassmannians as attracting sets of $\mathbb{C}^*$-fixed points which is used for example in Section 5.5.1 that there are only finitely many one-dimensional T-orbits in each cell and hence in the the whole quiver Grassmannian. The orbits can not leave the cells because the cells are T-stable. Moreover these one-dimensional orbits are in one to one correspondence with the fundamental mutations of the subquivers in the coefficient quiver of $M_{m,N}$. This is again a direct consequence of the parametrisation of the cells as attracting sets.

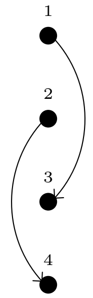
The labels of the edges are given by annihilators of the hyperplanes in the Lie algebra $\mathfrak{t}$ which correspond to the one-dimensional T-orbits. These vectors are obtained as follows. For the action of T take $\epsilon_i - \epsilon_j$ where $i \in [mN]$ is the index of the vertex in the coefficient quiver where the moved subsegment starts before the fundamental mutation of the coefficient quiver and $j \in [mN]$ is the index of the vertex where it starts after the movement. For the action of T_0 we have indices in $[m]$ with the additional information how many blocks of size $m \in [N]$ the starting point has moved down. With the identification of the different basis of $\mathbb{C}^{mN}$ we can compute the label $\epsilon_i - \epsilon_j + m\delta$ from the label for the action of T.

Everything as mentioned above is still conjectural but all computed examples suggest that it should hold in this generality. But unfortunately I was not able to work out all the details for the proofs before I had to submit this thesis. So I decided to restrict to one example for the explicit computations.

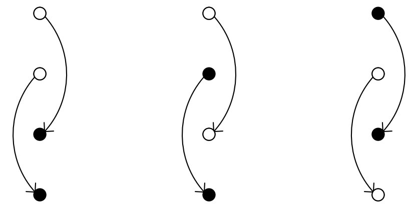
Example 7.4. For $x = y = 1, N = 2$, the quiver Grassmannian $Gr_2(A_2 \otimes \mathbb{C}^4)$ contains two-dimensional subrepresentations of the representation

$$M_{2,2} := \Big(\big(\mathbb{C}^4\big), \big(s_2\big)\Big)$$

which coefficient quiver is given by



The successor closed subquivers parametrising the cells of the quiver Grassmannian are of the form



These are the only possibilities for successor closed subquivers on two vertices because the other three possible subquivers one two vertices are not successor closed. The corresponding $\mathbb{C}^*$-fixed points are

$$p_0 = \mathrm{Span}\big(v_3, v_4\big), \quad p_1 = \mathrm{Span}\big(v_2, v_4\big), \quad p_2 = \mathrm{Span}\big(v_1, v_3\big)$$

where the index of p_i is equal to the dimension of the corresponding cells which is computed by counting the holes below the starting points of the segments in the coefficient quiver. From the description as attracting sets of the fixed points we obtain the subsequent description of the cells

$$\begin{aligned} c_0 &= \mathrm{Span}\big(v_3, v_4\big) = p_0 \\ c_1 &= \Big\{\mathrm{Span}\big(v_2 + a v_3, v_4\big) : a \in \mathbb{C}\Big\} \\ c_2 &= \Big\{\mathrm{Span}\big(v_1 + a v_2 + b v_4, v_3 + a v_4\big) : a, b \in \mathbb{C}\Big\}. \end{aligned}$$

The torus $T = \big(\mathbb{C}^*\big)^4$ acts on $v = \sum_{i=1}^4 \mu_i v_i \in \mathbb{C}^4$ as

$$\lambda.v = \sum_{i\in[2]} \lambda_{i+2(s_i-1)} \sum_{\ell=s_i}^{2} v_{i+2(\ell-1)}$$

where we used the identification $v_{i,\ell} = v_{i+m(\ell-1)}$ for the basis of $\mathbb{C}^{mN}$.

It is straight forward to check that the $\mathbb{C}^*$-fixed points and cells are T_0-fixed

$$\begin{aligned} \lambda.p_0 &= \mathrm{Span}\big(\lambda_3 v_3, \lambda_4 v_4\big) = \mathrm{Span}\big(v_3, v_4\big) = p_0 = c_0, \\ \lambda.p_1 &= \mathrm{Span}\big(\lambda_2 v_2, \lambda_2 v_4\big) = \mathrm{Span}\big(v_2, v_4\big) = p_1, \\ \lambda.p_2 &= \mathrm{Span}\big(\lambda_1 v_1, \lambda_1 v_3\big) = \mathrm{Span}\big(v_1, v_3\big) = p_2, \end{aligned}$$

$$\begin{aligned}
\lambda.c_1 &= \Big\{ \mathrm{Span}\big(\lambda_2 v_2 + \lambda_3 a v_3, \lambda_2 v_4\big) : a \in \mathbb{C} \Big\} \\
&= \Big\{ \mathrm{Span}\big(v_2 + \tilde{a} v_3, v_4\big) : \tilde{a} \in \mathbb{C} \Big\} = c_1 \\
\lambda.c_2 &= \Big\{ \mathrm{Span}\big(\lambda_1 v_1 + \lambda_2 a v_2 + \lambda_2 b v_4, \lambda_1 v_3 + \lambda_2 a v_4\big) : a \in \mathbb{C}^*, b \in \mathbb{C} \Big\} \\
&\quad \cup \Big\{ \mathrm{Span}\big(\lambda_1 v_1 + \lambda_4 b v_4, \lambda_1 v_3\big) : b \in \mathbb{C} \Big\} \\
&= \Big\{ \mathrm{Span}\big(v_1 + \tilde{a} v_2 + \tilde{b} v_4, v_3 + \tilde{a} v_4\big) : \tilde{a}, \tilde{b} \in \mathbb{C} \Big\} = c_2.
\end{aligned}$$

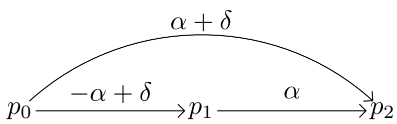
Accordingly the cell c_1 is a one-dimensional T-orbit and the point in its closure is p_0 such that there is an edge from p_0 to p_1 corresponding to this orbit and it is oriented towards p_1. The label of this edge is $\epsilon_3 - \epsilon_2$ since the effective action of the torus is given by λ_3/λ_2. For the embedding $T_0 \hookrightarrow T$ as defined above we obtain the effective action of $\gamma_1\gamma_0/\gamma_2$ which corresponds to the label $\epsilon_1 - \epsilon_2 + \delta$.

In the zero-dimensional cell there can not be any one-dimensional T-orbits and in the top-dimensional cell c_2 there are two one-dimensional T-orbits q_a and q_b corresponding to the cases $b = 0$ and $a = 0$. If both parameters are non-zero the corresponding T-orbit is two-dimensional. There can not be any other one-dimensional T-orbits in this quiver Grassmannian. Now we want to determine the two corresponding edges in the moment graph.

$$\begin{aligned}
\lambda.q_a &= \Big\{ \mathrm{Span}\big(\lambda_1 v_1 + \lambda_2 a v_2, \lambda_1 v_3 + \lambda_2 a v_4\big) : a \in \mathbb{C} \Big\} \\
&= \Big\{ \mathrm{Span}\big(v_1 + \frac{\lambda_2}{\lambda_1} a v_2, v_3 + \frac{\lambda_2}{\lambda_1} a v_4\big) : a \in \mathbb{C} \Big\} \\
\lambda.q_b &= \Big\{ \mathrm{Span}\big(\lambda_1 v_1 + \lambda_4 b v_4, \lambda_1 v_3\big) : b \in \mathbb{C} \Big\} \\
&= \Big\{ \mathrm{Span}\big(v_1 + \frac{\lambda_4}{\lambda_1} b v_4, v_3\big) : b \in \mathbb{C} \Big\}.
\end{aligned}$$

In the closure of q_a we have the point p_1 such that the corresponding edge is directed from p_1 to p_2. The label is given by $\epsilon_2 - \epsilon_1$ because the effective action is by λ_2/λ_1. For the embedding of T_0 in T we have the same label in this setting. The closure of q_b contains the point p_0 such that we obtain an edge directed from p_0 to p_2 and the label for the T-action is $\epsilon_4 - \epsilon_1$. For the action of T_0 we obtain the label $\epsilon_2 - \epsilon_1 + \delta$.

With $\alpha := \epsilon_2 - \epsilon_1$ the moment graph for the action of the torus T_0 on the quiver Grassmannian $\mathrm{Gr}_2(A_2 \otimes \mathbb{C}^2)$ is given as



The same graph with the same labels is computed using fundamental mutations and the procedure to label the edges as described above this example.

Moreover this is the moment graph as computed for the interval $[0, -\alpha]$ of $\hat{\mathcal{G}}^{par}$ for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$ by M. Lanini in [**56**, Example 4.2]. Observe that in her picture of the moment graph the zero-dimensional cell is in the middle, the cells on the left of it have even dimension and the cells on the right have odd dimension and as for the corresponding quiver Grassmannians there is exactly one cell of each dimension.

In the appendix of this article she shows that in this setting the structure sheaf and the BMP-sheaf are isomorphic. This implies that the T-equivariant intersection cohomology is equal to the ordinary T-equivariant cohomology and hence the intersection cohomology equals the ordinary cohomology [**11**, Theorem 1.6 and 1.7].

In bigger examples one might want to avoid drawing coefficient quivers. Recall that the cells in the quiver Grassmannian $\mathrm{Gr}_{xN}(M_{m,N})$ are parametrised by tuples $(k_i)_{i\in[m]}$ with $k_i \in \{0, 1, \ldots, N\}$ and $\sum_{i=1}^{m} k_i = xN$. But this parametrisation has the drawback that the notion of fundamental mutations and the corresponding labelling and orientation of the edges becomes harder to control.

Nevertheless it is possible to give a complete description of the moment graph in this setting. Let k and ℓ be two tuples parametrising cells in the quiver Grassmannian $\mathrm{Gr}_{xN}(M_{m,N})$. From the correspondence of one-dimensional orbits and fundamental mutations we obtain that they are connected by an edge if and only if there exists an pair of distinct indices $i, j \in [m]$ such that $k_i \neq \ell_i$, $k_j \neq \ell_j$ and $k_r = \ell_r$ for all $r \in [m] \setminus \{i, j\}$.

To determine the orientation and labelling of the edges let us assume that $i < j$. Choose $s \in \{i, j\}$ such that $k_s < \ell_s$. This index has to exist if k and ℓ are connected by an edge. Let t be the remaining index from $\{i, j\}$. Now we take $q \in \mathbb{Z}$ such that $k_s = \ell_t + q$ and it follows from the dimension of the representations in the quiver Grassmannian that $k_t = \ell_s - q$. We have to distinguish three different cases.

For $q = 0$ we orient the edge towards the tuple with the bigger i-th entry and label the edge by $\epsilon_j - \epsilon_i$. If $q < 0$ we orient the edge towards the tuple k and the edge is labelled by $\epsilon_s - \epsilon_t + q\delta$. For $q > 0$ we orient the edge towards the tuple ℓ and the label is given by $\epsilon_t - \epsilon_s + q\delta$. Analogous we can compute the labelling for the action of T.

It is straight forward to check that in the above example we obtain the same labelling as computed from the T_0-action or equivalently from the fundamental mutations of the coefficient quivers. The proof for the general setting is analogous to the computations in this example. It turns out that in all computed examples the moment graph and its labelling for the approximations of $\mathrm{Gr}(\widehat{\mathfrak{gl}}_n)$ has the same shape as the moment graph for certain finite intervals of $\hat{\mathcal{G}}^{par}$ for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_n$ which are of the same form as in the example above.

APPENDIX A

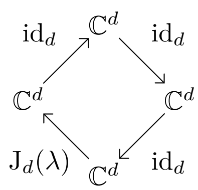
Examples: Quiver Grassmannians for the Equioriented Cycle

In this appendix we give some examples of quiver Grassmannians for the equioriented cycle which do not satisfy all the properties of the class of quiver Grassmannians studied for the main part of this thesis. These examples illustrate why the restrictions on the studied class of quiver Grassmannians were made. Moreover we give counterexamples for some geometric properties which are obtained for a similar class of quiver Grassmannians for Dynkin quivers.

Example A.1. Let $n = 4$ and

$$M := U_1(4) \oplus U_2(4) \oplus U_3(4) \oplus U_4(4), \quad \tilde{M}_\lambda := \bigoplus_{i \in \mathbb{Z}_4} U_i(1;\lambda)$$

and $\mathbf{e} := {}^t(2,2,2,2)$ where $U_i(d;\lambda)$ denotes the representation



of the equioriented cycle Δ_4 where the map from the vertex i to the vertex $i+1$ is given by $\mathrm{J}_d(\lambda)$, i.e. the Jordanblock of size d with the eigenvalue λ. Then for $\lambda \in \mathbb{C}^*$ we obtain

$$\mathrm{Gr}_2(4) \cong \mathrm{Gr}_{\mathbf{e}}^{\Delta_4}(\tilde{M}_\lambda) \quad \text{and define} \quad \mathrm{Gr}_2^a(4) := \mathrm{Gr}_{\mathbf{e}}^{\Delta_4}(M)$$

which we call the degenerate Grassmannian.

This constructions generalise to arbitrary $n, k \in \mathbb{N}$. The classical Grassmannian has dimension $k(n-k)$. From the formula for the dimension of quiver Grassmannians for the cycle Δ_n as developed in Lemma 3.22 we obtain that this equals the dimension of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(M)$ for the quiver representation

$$M := \bigoplus_{i \in \mathbb{Z}_n} U_i(n)$$

and the dimension vector $\mathbf{e} := {}^t(k,\dots,k)$. Following Lemma 3.23 the number of irreducible components of this quiver Grassmannian is given by $\binom{n}{k}$.

For $\lambda = 0$ we obtain $\tilde{M}_0 = M$ such that the quiver Grassmannians $\mathrm{Gr}_{\mathbf{e}}^{\Delta_n}(\tilde{M}_\lambda)$ are a family over $\mathbb{C}$ with special fibre $\mathrm{Gr}_k^a(n)$ and generic fibre isomorphic to $\mathrm{Gr}_k(n)$. A degeneration is called flat if the morphism above is flat. For this it is necessary that the fibre dimension is constant. This is satisfied in the setting of this example.

For a certain class of representations of Dynkin quivers there exists a unique (up to isomorphism) representation with the same dimension vector such that the quiver Grassmannian corresponding to the first representation is a flat degeneration of the quiver Grassmannian corresponding to the second representation [**20**, Theorem 3.2].

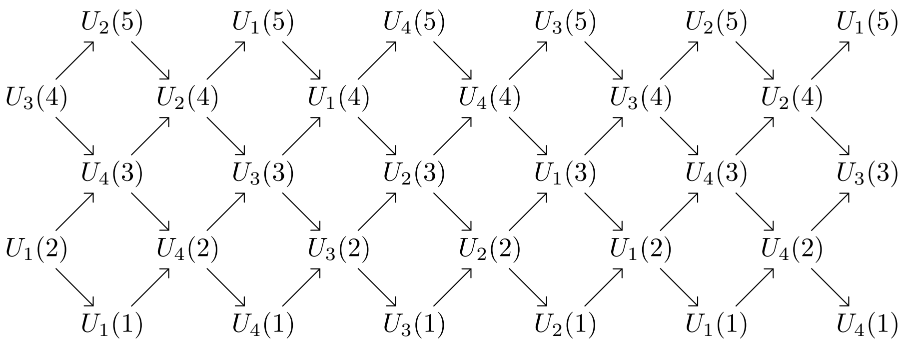
The quiver Grassmannians studied in this thesis are based on representations which live in a generalisation of the class of quiver representations studied in [**20**]. Hence it is natural to ask if there exists a similar statement about flat degenerations for representations of the cycle Δ_n. In the Dynkin setting the statement mentioned above can be applied to show that the degeneration of the classical flag variety is flat. For the affine flag variety and the affine Grassmannian we would have some similar statement if we could show it for the quiver Grassmannians approximating them. But as shown the Chapter 5 and Chapter 6 we do not get the equidimensionality of the approximations. Accordingly it is not possible to prove the flatness with methods from quiver theory and we would have to apply different theory as done for example in [**30**] to prove the flatness for the degeneration of the affine Grassmannian. But in this thesis we restrict us to methods from the study of modules over finite dimensional algebras.

A.1. Strata outside the Closure of the Stratum of X

Let Xbe a projective and Y be a injective representation of a Dynkin quiver Q. Then the closure of the stratum of X is the whole quiver Grassmannian $\mathrm{Gr}_{\mathbf{dim}\,X}(X \oplus Y)$ [**20**, Theorem 1.1]. For projective and injective representations of the equioriented cycle it follows from Lemma 3.23 that the corresponding quiver Grassmannians are irreducible if and only if X or Y is trivial. If we give up the restriction that the summands of X and Y all have the same length there are a few more cases. All in all irreducibility of quiver Grassmannians for the cycle is much more rare than in the Dynkin setting. Moreover there are cases where the dimension of the quiver Grassmannian is strictly bigger than the dimension of the stratum of X.

For $N \neq \omega \cdot n$ the subsequent examples contradict the statement about the dimension and the parametrisation of the irreducible components in the Grassmannians as proven for $N = \omega \cdot n$ in Lemma 3.23.

Example A.2. Let $n = 4, N = 5$ and $X := U_3(5) \oplus U_4(5)$ be a bounded projective representation of the equioriented cycle Δ_4. In this setting the part of the Auslander Reiten Quiver with the nilpotent representations of maximal length 5 is given as



where points with the same label are identified such that we have meshes on a cylinder. Here we do not want to go into the details how Auslander Reiten Quivers are defined in general and how they are obtained for the equioriented cycle. The definition and some possible ways of construction are given in the book by R. Schiffler [**66**, Chapter 1.5, Chapter 3] and the book by I. Assem, D. Simson and A. Skowronski [**2**, Chapter IV]. The dimension vector of X computes as

$$\mathbf{e} := \mathbf{dim}\, X = {}^t(1,1,2,1) + {}^t(1,1,1,2) = {}^t(2,2,3,3)$$

and $\dim \mathrm{GL}_{\mathbf{e}} = 2^2 + 2^2 + 3^2 + 3^2 = 4+4+9+9 = 26$. The dimension of the space of endomorphisms is given as

$$\dim \mathrm{Hom}_{\Delta_4}\big(X, X\big) = (2+1) + (1+2) = 6$$

which can be computed using the word combinatorics as in Proposition 3.16. Define $U := U_2(4) \oplus U_2(4) \oplus U_3(2)$. Then

$$\dim \mathrm{Hom}_{\Delta_4}\big(U, U\big) = (1+0+0) + (0+1+1) + (0+1+1) = 5.$$

For $Y := U_1(5) \oplus U_1(5)$ we obtain $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_4}(X \oplus Y)$ from the shape of the indecomposable embeddings as described in Proposition 3.19 or from the structure of the Auslander Reiten Quiver. The dimension of the strata compute as

$$\begin{aligned}
\dim \mathcal{S}_U &= \dim \mathrm{Hom}_{\Delta_4}\big(U, X\big) + \dim \mathrm{Hom}_{\Delta_4}\big(U, Y\big) - \dim \mathrm{Hom}_{\Delta_4}\big(U, U\big)\\
&= \big((1+1)+(1+1)+(1+1)\big) + \big((0+0)+(1+1)+(1+1)\big) - 5 = 5\\
\dim \mathcal{S}_X &= \dim \mathrm{Hom}_{\Delta_4}\big(X, X\big) + \dim \mathrm{Hom}_{\Delta_4}\big(X, Y\big) - \dim \mathrm{Hom}_{\Delta_4}\big(X, X\big)\\
&= \big((1+1)+(1+1)\big) = 4.
\end{aligned}$$

The indecomposable summands of X and Y have all the same length. Hence we can apply Theorem 2.3 in order to compute the closures of the strata in the quiver Grassmannian using the results by G. Kempken about the orbit closures in the variety of quiver representations. We obtain that X is not included in the closure of the stratum of U. Accordingly the Grassmannian is not irreducible and its dimension strictly bigger than the dimension of the irreducible component of X. For any Dynkin quiver Q the closure of the stratum of a projective representation X would already be the whole Grassmannian

$$\mathrm{Gr}_{\mathbf{e}}(X \oplus Y)$$

where Y is a injective representation of Q and $\mathbf{e}$ is the dimension vector of X [**20**, Theorem 1.1].

Example A.3. Let $n = 7, N = 4$ and define

$$\begin{aligned}
X &:= U_3(4) \oplus U_4(4) \oplus U_7(4),\\
Y &:= U_1(4) \oplus U_2(4) \oplus U_5(4) \quad \text{and}\\
U &:= U_2(4) \oplus U_5(4) \oplus U_3(2) \oplus U_6(2).
\end{aligned}$$

Then $U \in \mathrm{Gr}_{\mathbf{e}}^{\Delta_7}(X \oplus Y)$ where $\mathbf{e} := \mathbf{dim}\, X$ since the coefficient quiver of U is a subquiver in the coefficient quiver of $X \oplus Y$ and has the right dimension vector.

Using word combinatorics or a computer algebra system we obtain

$$\begin{aligned}
\dim \mathcal{S}_U &= \dim \operatorname{Hom}_{\Delta_7}\big(U, X\big) + \dim \operatorname{Hom}_{\Delta_7}\big(U, Y\big) - \dim \operatorname{Hom}_{\Delta_7}\big(U, U\big)\\
&= \big((1+0+0) + (0+0+0) + (1+1+0) + (0+1+1)\big)\\
&\quad + \big((0+0+1) + (1+1+0) + (0+0+1) + (1+1+0)\big)\\
&\quad - \big((1+0+0+0) + (0+1+0+0) + (0+0+1+0) + (0+0+1+1)\big)\\
&= 6 + 5 - 5 = 6\\
\dim \mathcal{S}_X &= \dim \operatorname{Hom}_{\Delta_7}\big(X, X\big) + \dim \operatorname{Hom}_{\Delta_7}\big(X, Y\big) - \dim \operatorname{Hom}_{\Delta_7}\big(X, X\big)\\
&= \big((1+1+0) + (1+1+0) + (0+0+1)\big) = 5.
\end{aligned}$$

Again we can check that X is not contained in the closure of the stratum of U. Thus this Grassmannian is not irreducible and its dimension is strictly bigger than the dimension of the stratum of X.

A.2. Irreducible Components of different Dimension

Even if the dimension of the Grassmannian is the same as the dimension of the stratum of X the other irreducible components of the Grassmannian do not have to be of the same dimension.

Example A.4. Let $n = 3, N = 2$. Define $X := U_2(2) \oplus U_2(2) \oplus U_3(2)$, $Y := U_1(2) \oplus U_1(2) \oplus U_2(2)$ and $\mathbf{e} := \mathbf{dim}\, X = {}^t(1, 2, 3)$. In this setting we obtain the Auslander Reiten Quiver

$$\begin{array}{ccccccccccc}
U_1(2) & & U_3(2) & & U_2(2) & & U_1(2) & & U_3(2) & & \\
& \searrow & & \nearrow \searrow & & \nearrow \searrow & & \nearrow \searrow & & \nearrow \searrow & \\
& U_1(1) & & U_3(1) & & U_2(1) & & U_1(1) & & U_3(1) &
\end{array}$$

The isomorphism classes of subrepresentations of $X \oplus Y$ with dimension vector $\mathbf{e}$ are described by the following representatives:

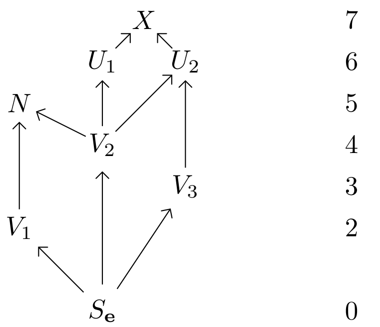
$$\begin{aligned}
X &:= U_2(2) \oplus U_2(2) \oplus U_3(2) & V_1 &:= U_1(2) \oplus S_2 \oplus S_3 \oplus S_3 \oplus S_3\\
U_1 &:= U_2(2) \oplus U_3(2) \oplus S_2 \oplus S_3 & V_2 &:= U_2(2) \oplus S_1 \oplus S_2 \oplus S_3 \oplus S_3\\
U_2 &:= U_2(2) \oplus U_2(2) \oplus S_1 \oplus S_2 & V_3 &:= U_3(2) \oplus S_2 \oplus S_2 \oplus S_3 \oplus S_3\\
N &:= U_1(2) \oplus U_2(2) \oplus S_3 \oplus S_3 & S_{\mathbf{e}} &:= S_1 \oplus S_2 \oplus S_2 \oplus S_3 \oplus S_3 \oplus S_3
\end{aligned}$$

We can compute them using Proposition 3.19 about the structure of the indecomposable embeddings or using a computer algebra system. For the equioriented cycle we have the following equivalence

$$\mathcal{S}_U \subset \overline{\mathcal{S}_V} \Leftrightarrow \mathcal{O}_U \subset \overline{\mathcal{O}_V}$$

if the strata live in the quiver Grassmannian of a representation M where all indecomposable summands have the same length. In general we only have the implication $U \in \overline{\mathcal{S}_V} \Rightarrow \mathcal{O}_U \subset \overline{\mathcal{O}_V}$ [**41**, Theorem 3.4.1]. The closures of orbits in the variety of quiver representations were studied in the thesis of G. Kempken [**48**]. We can use Theorem 3.13 to compute the dimension of the higher dimensional strata from the dimension of the strata of $S_{\mathbf{e}}$ and its dimension could be computed using the corresponding subquivers of the coefficient quiver of $X \oplus Y$. In the following diagram we collect information about closures and the dimension of the strata

Representatives: $\quad\quad \dim \mathcal{S}$:



where we draw $U \mapsto V$ if $\mathcal{O}_U \subset \overline{\mathcal{O}_V}$. The corresponding quiver Grassmannian therefore has two irreducible components of distinct dimension.

The above examples should illustrate that in the general setup a lot of different problems could arise. That is the reason why we we restrict us to the case $N = \omega \cdot n$ for the main part of this work.

A.3. Intersection of Irreducible Components

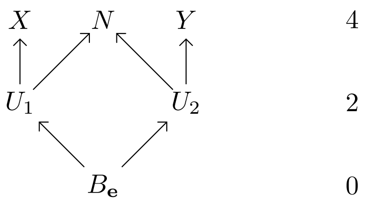
With the parametrisation of the irreducible components of the quiver Grassmannians obtained for $N = \omega \cdot n$ we can go one step further and examine the structure of the intersection of two irreducible components. Namely we ask if the section of two irreducible components is again irreducible and if the codimension of the intersection is minimal within the components.

Example A.5. Let $n = N = 3$. Define $X := U_1(3) \oplus U_1(3)$, $Y := U_2(3) \oplus U_2(3)$ and $\mathbf{e} := \mathbf{dim}\, X$. The isomorphism classes of subrepresentations of $X \oplus Y$ with dimension vector $\mathbf{e}$ are described by the following representatives:

$$U_1 := U_1(3) \oplus U_2(2) \oplus S_1 \qquad X := U_1(3) \oplus U_1(3)$$
$$U_2 := U_2(3) \oplus U_2(2) \oplus S_1 \qquad N := U_1(3) \oplus U_2(3)$$
$$B_{\mathbf{e}} := U_2(2) \oplus U_2(2) \oplus S_1 \oplus S_1 \qquad Y := U_2(3) \oplus U_2(3)$$

We obtain the subsequent diagram

Representatives: $\quad\quad \dim \mathcal{S}$:



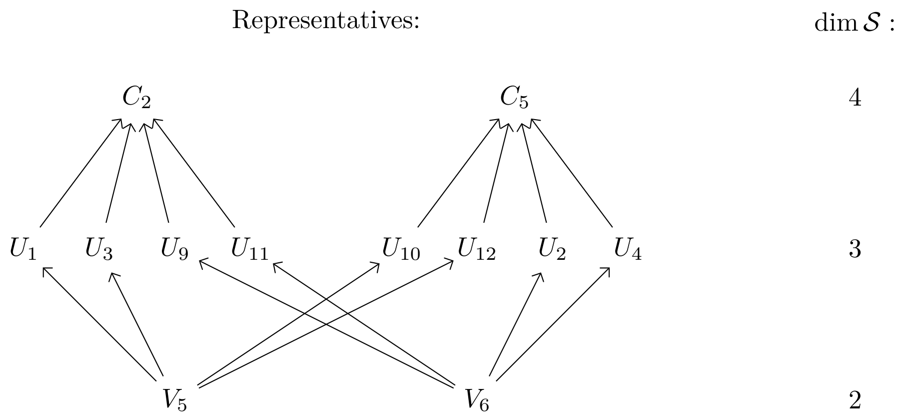
The section $\overline{\mathcal{S}_X} \cap \overline{\mathcal{S}_Y}$ is given by $\overline{\mathcal{S}_{B_{\mathbf{e}}}}$ with codimension four and $\overline{\mathcal{S}_{U_1}}$ is the section $\overline{\mathcal{S}_X} \cap \overline{\mathcal{S}_N}$ which has codimension two.

This example shows that there can not be a general statement about the codimension of the section of two irreducible components. But it suggests that the answer to the first question might be positive.

EXAMPLE A.6. Let $n = N = 4$. Define $M := U_1(4) \oplus U_2(4) \oplus U_3(4) \oplus U_4(4)$ and $\mathbf{e} := (2, 2, 2, 2)$. The isomorphism classes of subrepresentations of M with dimension vector $\mathbf{e}$ are described by the following representatives:

$$
\begin{aligned}
C_1 &:= U_1(4) \oplus U_2(4) & U_1 &:= U_1(4) \oplus U_3(3) \oplus U_2(1)\\
C_2 &:= U_1(4) \oplus U_3(4) & U_2 &:= U_2(4) \oplus U_4(3) \oplus U_3(1)\\
C_3 &:= U_2(4) \oplus U_3(4) & U_3 &:= U_3(4) \oplus U_1(3) \oplus U_4(1)\\
C_4 &:= U_1(4) \oplus U_4(4) & U_4 &:= U_4(4) \oplus U_2(3) \oplus U_1(1)\\
C_5 &:= U_2(4) \oplus U_4(4) & U_5 &:= U_1(4) \oplus U_2(2) \oplus U_4(2)\\
C_6 &:= U_3(4) \oplus U_4(4) & U_6 &:= U_2(4) \oplus U_1(2) \oplus U_3(2)\\
V_1 &:= U_1(3) \oplus U_2(3) \oplus U_4(2) & U_7 &:= U_3(4) \oplus U_2(2) \oplus U_4(2)\\
V_2 &:= U_2(3) \oplus U_3(3) \oplus U_1(2) & U_8 &:= U_4(4) \oplus U_1(2) \oplus U_3(2)\\
V_3 &:= U_3(3) \oplus U_4(3) \oplus U_2(2) & U_9 &:= U_1(4) \oplus U_4(3) \oplus U_3(1)\\
V_4 &:= U_1(3) \oplus U_4(3) \oplus U_3(2) & U_{10} &:= U_2(4) \oplus U_1(3) \oplus U_4(1)\\
V_5 &:= U_1(3) \oplus U_3(3) \oplus U_2(1) & U_{11} &:= U_3(4) \oplus U_2(3) \oplus U_1(1)\\
V_6 &:= U_2(3) \oplus U_4(3) \oplus U_1(1) & U_{12} &:= U_4(4) \oplus U_3(3) \oplus U_2(1)\\
V_7 &:= U_1(4) \oplus U_4(2) \oplus U_2(1) \oplus U_3(1) & W_1 &:= U_1(3) \oplus U_3(2) \oplus U_4(2) \oplus U_2(1)\\
V_8 &:= U_2(4) \oplus U_1(2) \oplus U_3(1) \oplus U_4(1) & W_2 &:= U_2(3) \oplus U_4(2) \oplus U_1(2) \oplus U_3(1)\\
V_9 &:= U_3(4) \oplus U_2(2) \oplus U_1(1) \oplus U_4(1) & W_3 &:= U_3(3) \oplus U_1(2) \oplus U_2(2) \oplus U_4(1)\\
V_{10} &:= U_4(4) \oplus U_3(2) \oplus U_1(1) \oplus U_2(1) & W_4 &:= U_4(3) \oplus U_2(2) \oplus U_3(2) \oplus U_1(1)\\
B_{\mathbf{e}} &:= U_1(2) \oplus U_2(2) \oplus U_3(2) \oplus U_4(2)
\end{aligned}
$$

All representations labelled with C have four-dimensional strata, U's correspond to three-dimensional strata, V's to two-dimensional, W's to one-dimensional and the stratum of $B_{\mathbf{e}}$ is zero-dimensional. In the following diagram we show representatives for the strata of codimension one included in the closures of the strata of C_2 and C_5. For codimension two we display representatives only for some of the strata included in the closures.



Accordingly there is no section in codimension one and in codimension two the section has at least two components. This shows that the section of two irreducible components in general is not irreducible.

A.4. Orbits and Strata of different Codimension

For all examples above we stayed in the setting where all indecomposable summands of X and Y have the same length. Thus it is possible to apply Theorem 2.3 and we can examine the orbits in the variety of quiver representations in order to understand the strata in the quiver Grassmannian. We finish this appendix with an example where this correspondence fails.

Example A.7. Let $M := A_6 \oplus A_3 \otimes \mathbb{C}^2$ be a representation of the loop quiver Δ_1. The dimension of the vector space corresponding to M is 12 and we consider the quiver Grassmannian $\mathrm{Gr}_6^{A_6}(M)$. It contains the isomorphism classes of the representations

$$U_0 := A_3 \otimes \mathbb{C}^2, \quad U_1 := A_4 \oplus A_2, \quad U_2 := A_5 \oplus A_1, \quad U_3 := A_6.$$

Using Proposition 5.1 the dimension of their strata computes as

$$\dim \mathrm{Hom}_{\Delta_1}\big(U_0, M\big) = 2 \cdot 3 + 2 \cdot 2 \cdot 3 = 18$$
$$\dim \mathrm{Hom}_{\Delta_1}\big(U_0, U_0\big) = 2 \cdot 2 \cdot 3 = 12$$
$$\dim \mathcal{S}_{U_0} = \dim \mathrm{Hom}_{\Delta_1}\big(U_0, M\big) - \dim \mathrm{Hom}_{\Delta_1}\big(U_0, U_0\big) = 18 - 12 = 6$$

$$\dim \mathrm{Hom}_{\Delta_1}\big(U_1, M\big) = 4 + 2 \cdot 3 + 2 + 2 \cdot 2 = 16$$
$$\dim \mathrm{Hom}_{\Delta_1}\big(U_1, U_1\big) = 4 + 2 + 2 + 2 = 10$$
$$\dim \mathcal{S}_{U_1} = \dim \mathrm{Hom}_{\Delta_1}\big(U_1, M\big) - \dim \mathrm{Hom}_{\Delta_1}\big(U_1, U_1\big) = 16 - 10 = 6$$

$$\dim \mathrm{Hom}_{\Delta_1}\big(U_2, M\big) = 5 + 2 \cdot 3 + 1 + 2 \cdot 1 = 14$$
$$\dim \mathrm{Hom}_{\Delta_1}\big(U_2, U_2\big) = 5 + 1 + 1 + 1 = 8$$
$$\dim \mathcal{S}_{U_2} = \dim \mathrm{Hom}_{\Delta_1}\big(U_2, M\big) - \dim \mathrm{Hom}_{\Delta_1}\big(U_2, U_2\big) = 14 - 8 = 6$$

$$\dim \mathrm{Hom}_{\Delta_1}\big(U_3, M\big) = 6 + 2 \cdot 3 = 12$$
$$\dim \mathrm{Hom}_{\Delta_1}\big(U_3, U_3\big) = 6$$
$$\dim \mathcal{S}_{U_3} = \dim \mathrm{Hom}_{\Delta_1}\big(U_3, M\big) - \dim \mathrm{Hom}_{\Delta_1}\big(U_3, U_3\big) = 12 - 6 = 6$$

whereas their orbit dimensions in the variety of quiver representations is given as

$$\dim \mathcal{O}_{U_3} = 30, \quad \dim \mathcal{O}_{U_2} = 28, \quad \dim \mathcal{O}_{U_1} = 26, \quad \dim \mathcal{O}_{U_0} = 24$$

because $\dim \mathrm{GL}_e = 6 \cdot 6 = 36$. Moreover it follows from G. Kempkens results that

$$\mathcal{O}_{U_{i-1}} \subset \overline{\mathcal{O}_{U_i}} \quad \text{for all } i \in [3].$$

This quiver Grassmannian provides a finite approximation of a partial degenerate affine Grassmannian and its Poincaré polynomial and Euler characteristic is computed in Example C.28. Conjecture 5.32 about the dimension and irreducible components of the approximations of the partial degenerate affine Grassmannian is based on the assumption that the strata as described above are always the strata of biggest dimension.

APPENDIX B

Computer Programs

In this appendix we present programs for the computation of the Euler characteristic and the Poincaré polynomial of the approximations of the affine flag variety and the affine Grassmannian as well as their partial degenerations. These programs are based on the parametrisation of the cells by successor closed subquivers in the coefficient quiver of the quiver representation whose quiver Grassmannian provides the approximation. The dimension function for the cells is based on the observation that the dimension of a cell is given by the number of holes below the starting points of the segments in the coefficient quiver which parametrise the cell.

B.1. For Approximations of Partial-Degenerate Affine Flag Varieties

```
#
# Program to compute the Euler characteristics and Poincare polynomials for approximations of the affine Flag variety and its partial degenerations
#

# Parameters ***************************************************

# n = index of gl_n
# w = parameter such that 2wn = length of longest allowed indecomposable nilpotet representation of the equioriented cycle
# p = vector containing entries 1 or 0 parametrising the maps f_i : U_i --> U_i+1
#     f_i = id if p_i = 1 and f_i = pr_i if p_i = 0

# the multiplicities of the projective and injective representations is one for both and the dimension vector of subrepresentations is given by e = (wn, ... , wn)

Z = IntegerRing()

# Basics ***************************************************

```

```
# seting up the dimension vectors of the indecomposable representations for w = 1
# sufficient to compute the dimension vectors of longer indecomposable representations
# ID = identity_matrix(n)
def index_set(ID, i):
    n = len(ID[0])
    v = 0*ID[0]
    D_i = [ v ]
    for j in xrange(n):
        k = (i-j+n-1) % n
        v += ID[k]
        D_i += [ v ]

    return D_i


# dimension vector of the indecomposable representation of length k embedding into U_i(2wn)
def dim_vec(D_i,k):
    vec = D_i[0]
    n = len(vec)
    r = k%n
    for j in xrange(1,floor(k/n)+1):
        vec += D_i[n]

    vec += D_i[r]
    return vec


# check vector for positivity component wise
def geq_null(v):
    positive = true
    n = len(v)
    for i in xrange(n):
        if v[i] < 0:
            positive = false
            break

    return positive


# check if vector is zero vector
def eq_null(v):
    zero = true
    n = len(v)
    for i in xrange(n):
        if not v[i] == 0:
            zero = false
```

```
            break

        return zero

# Main Computation
    ************************************************

# computing all cells of the quiver Grassmannian
def cell_setup(p,w):
    n = len(p)
    ID = identity_matrix(n)
    null = 0*ID[0]

    # the dim vec of sub reps
    e = []
    for k in xrange(n):
        e += [w*n]

    # the dim vecs for w=1
    D = [ ]
    for i in xrange(n):
        D_i = index_set(ID,i)
        D += [ D_i ]

    # vector containing the maximal length for each segment
    max_length = [ ]
    # data parametrising candidates for cells
    t_tup = [ [] , vector(e) ]
    for i in xrange(n):
        for k in xrange(2-p[i]):
            max_length += [ (1+p[i])*w*n ]
            t_tup[0] += [0]

    # dimension vector of M
    dim = 2*vector(e)

    # number of segments
    n_seg = len(max_length)

    # computing the actual cells
    cells = []
    tupel_tmp = [ t_tup ]
    rest_dim = dim
    counter = 0
    for i in xrange(n):
        for k in xrange(2-p[i]):
            k_max = list(max_length)[counter]
            rest_dim -= vector(list(dim_vec(D[i],k_max)))
            for temp in tupel_tmp:
```

```
          if temp[0][counter] == 0:
            for l in xrange(k_max):
              # computing new candidate
              r = ( vector(list(temp[1])) - vector(list(dim_vec(D[i],l+1))) )
              if geq_null(r) == true:
                if eq_null(r) == true:
                  # we have found a cell!
                  new_cell = list(temp[0])
                  new_cell[counter] = (l+1)
                  cells += [ new_cell ]
                elif geq_null(rest_dim - r) == true:
                  # this can still become a cell
                  new_tup = [ list(temp[0]) , vector(list(e)) ]
                  new_tup[0][counter] = (l+1)
                  new_tup[1] = vector(list(r))
                  tupel_tmp += [ new_tup ]
        counter += 1
        sys.stdout.write('\r')
        sys.stdout.write('Setting up problem data: %d / %d' % (counter,n_seg))
        sys.stdout.flush()

    sys.stdout.write('\r')
    sys.stdout.write('                                                        \n')

    sys.stdout.write('\r')
    sys.stdout.write('   chi_e(M) = %d' % (len(cells)) )
    sys.stdout.write('\n')

    return cells


# function to compute the dimension of a cell in the Grassmannian from the corresponding subquiver in the coefficient quiver of M
def cell_dim(cell,w,p):
    dim = 0
    n = len(p)

    # data type for the cells
    # express them as segments in the coefficient quiver
    cell_mat = []
    for j in xrange(2*w*n):
      cell_mat += [ [] ]
      for i in xrange(n):
        cell_mat[j] += [1]
```

```
    # writing the segments in the matrix
    counter = 0
    starting_points = []
    for i in xrange(n):
      for k in xrange(2-p[i]):
        # the length of the current segment
        length = cell[counter]
        if length > 0:
          # position of the endpoint
          t = (i)%n
          # ht of the endpoint of the segment
          s = ( 2*w*n - k - 1 + (2-p[t]) )
          for step in xrange(length):
            # position at current step
            t = (i-step)%n
            # ht at current step
            if not ( p[t] == 0 and step >= w*n ):
              s -= (2-p[t])
            # marking the point corresponding to the
                segment
            cell_mat[s][t] = 0
          # collecting information on the starting points
          starting_points += [ [s,t] ]
        counter += 1

    # counting the holes i.e. ones below the starting
        points
    for starting_point in starting_points:
      start_ht = starting_point[0]
      start_index = starting_point[1]
      for below in xrange(start_ht,2*w*n):
        dim += cell_mat[below][start_index]

    return dim, cell_mat


# computing the Poincare polynomial of the quiver
    Grassmannian from the cells
def poincare_poly(p,w):
    tups = []
    coeffs = [ 0 ]
    cells = cell_setup(p,w)
    length_1 = len(cells)
    counter = 0
    for cell in cells:
      index, cell_mat = cell_dim(cell,w,p)
      tups +=[ [cell, cell_mat, index] ]
      length = len(coeffs)
```

```
      if (index + 1) > length:
        for k in xrange(index + 1 - length):
          coeffs += [ 0 ]
      coeffs[index] += 1
      counter += 1
      sys.stdout.write('\r')
      sys.stdout.write('Computing polynomial: %d / %d' %
          (counter,length_1))
      sys.stdout.flush()

    sys.stdout.write('\r')
    sys.stdout.write('

      \n')
    sys.stdout.write('    p_e,M(t) = ')
    for i in xrange(len(coeffs)):
      sys.stdout.write('%d*t^%d' %
          (coeffs[len(coeffs)-i-1],len(coeffs)-i-1))
      if i < (len(coeffs) - 1):
        sys.stdout.write(' + ')

    sys.stdout.write('\n')
    sys.stdout.write('\n')

    return tups
```

B.2. For Approximations of Partial-Degenerate Affine Grassmannians

```
#
# Program to compute the Euler characteristics and
    Poincare polynomials for approximations of the affine
    Grassmannian and its partial degenerations
#

# Parameters
    **************************************************

# n = index of gl_n
# N = parameter such that 2*N = length of longest allowed
    indecomposable nilpotet representation of the loop
# p = vector containing entries 1 or 0 parametrising the
    map f : U_0 --> U_0

# the multiplicities of the repesentations depend on n and
    the dimension vector of subrepresentations is given by
    nN

Z = IntegerRing()
```

```
# Main Computation
    ****************************************************

# computing all cells of the quiver Grassmannian
def setup_cells(p,N):
    n = len(p)
    cells = [ ]
    dim = 2*n*N

    # vector containing the maximal length for each segment
    max_length = [ ]
    # data parametrising candidates for cells
    t_tup = [ [] , [n*N] ]
    for i in xrange(n):
      for k in xrange(2-p[i]):
        max_length += [ (1+p[i])*N ]
        t_tup[0] += [0]

    # number of segments
    n_seg = len(max_length)

    # computing the actual cells
    tupel_tmp = [ t_tup ]
    rest_length = dim
    for j in xrange(n_seg):
      rest_length -= max_length[j]
      for temp in tupel_tmp:
        if temp[0][j] == 0:
          k_max = min(list(temp[1])[0],list(max_length)[j])
          for k in xrange(k_max):
            # computing new candidate
            r = ( list(temp[1])[0] - k - 1 )
            if r >= 0:
              if r == 0:
                # we have found a cell!
                new_cell = list(temp[0])
                new_cell[j] = (k+1)
                cells += [ new_cell ]
              elif r <= rest_length:
                # this can still become a cell
                new_tup = [ list(temp[0]) , list(temp[1]) ]
                new_tup[0][j] = (k+1)
                new_tup[1][0] -= (k+1)
                tupel_tmp += [ new_tup ]
      sys.stdout.write('\r')
      sys.stdout.write('Setting up problem data: %d / %d'
         % (j+1,n_seg))
      sys.stdout.flush()
```

```
    sys.stdout.write('\r')
    sys.stdout.write('

        \n')

    sys.stdout.write('\r')
    sys.stdout.write('   chi_e(M) = %d\n' % (len(cells)) )
    sys.stdout.write(' \n')

    return cells, max_length



# function to compute the dimension of a cell in the
    Grassmannian from the corresponding subquiver in the
    coefficient quiver of M
def cell_dim(cell, max_length):
    dim = 0
    ht_tupel = []
    n = len(cell)

    # data type for the cells
    # express them as segments in the coefficient quiver
        and count the holes below the starting points of
        the subsegments
    for i in xrange(n):
      ht_tupel += [0]
      if not cell[i] == 0:
        for j in xrange(n):
          if (j > i):
            ht_tupel[i] += ( min(cell[i], max_length[j]) -
                min(cell[i], cell[j]) )
          elif ( j < i ):
            ht_tupel[i] += ( min(cell[i]-1, max_length[j])
                - min(cell[i]-1, cell[j]) )

    for ht in ht_tupel:
      dim += ht

    return dim


# computing the Poincare polynomial of the quiver
    Grassmannian from the cells
def poincare_poly(p,N):
    tups = []
    coeffs = [ 0 ]
    dim_Grass = 0
```

```
    cells, max_length = setup_cells(p,N)
    length_1 = len(cells)
    counter = 0
    for cell in cells:
      index = cell_dim(cell, max_length)
      tups +=[ [cell, index] ]
      length = len(coeffs)
      if (index + 1) > length:
        for k in xrange(index + 1 - length):
          coeffs += [ 0 ]
      coeffs[index] += 1
      counter += 1
      sys.stdout.write('\r')
      sys.stdout.write('Computing polynomial: %d / %d' %
          (counter,length_1))
      sys.stdout.flush()

    sys.stdout.write('\r')
    sys.stdout.write('

       \n')
    sys.stdout.write('    p_e,M(t) = ')
    for i in xrange(len(coeffs)):
      sys.stdout.write('%d*t^%d' %
          (coeffs[len(coeffs)-i-1],len(coeffs)-i-1))
      if i < (len(coeffs) - 1):
        sys.stdout.write(' + ')

    sys.stdout.write('\n')
    sys.stdout.write('\n')

    return tups
```

APPENDIX C

Euler Characteristics and Poincaré Polynomials

In this appendix we collect Euler characteristics and Poincaré polynomials of some approximations of partial degenerate affine flag varieties and affine Grassmannians. Everything is computed using the programs as presented in Appendix B. Because of the limits in computational power and memory of the used machine it was not possible to handle the computations for bigger values of n or N. The most time and memory consuming part of the computation was the setup of the data for the cells. Optimising this part of the program could yield a much faster and more efficient computation. But for our purpose the computed data was sufficient such that we did not pursue this goal.

C.1. For Approximations of Partial-Degenerate Affine Flag Varieties

Cyclic permutations of the vector $\mathbf{c} \in \mathbb{Z}^n$ do not change the Poincaré polynomial because of the symmetry of the oriented cycle. Hence we only give the Poincaré polynomial for one representative of the isomorphism class of partial degenerations.

C.1.1. Poincaré Polynomials for n=1.

Example C.1. For $\omega \in \mathbb{N}$ and $n = 1$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\chi^a_{\omega,1} = \omega + 1$$

$$p^a_{\omega,1}(q) = \sum_{k=0}^{\omega} q^k$$

Example C.2. For $\omega \in \mathbb{N}$ and $n = 1$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\chi_{\omega,1} = 1$$

$$p_{\omega,1}(q) = q^0$$

C.1.2. Poincaré Polynomials for n=2.

EXAMPLE C.3. For $\omega \in [6]$ and $n = 2$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^a_\omega(\widehat{\mathfrak{gl}}_n)$ are given by

$$\chi^a_{1,2} = 15$$
$$p^a_{1,2}(q) = 3q^4 + 4q^3 + 5q^2 + 2q^1 + 1q^0$$
$$\chi^a_{2,2} = 65$$
$$p^a_{2,2}(q) = 3q^8 + 6q^7 + 13q^6 + 14q^5 + 13q^4 + 8q^3 + 5q^2 + 2q^1 + 1q^0$$
$$\chi^a_{3,2} = 175$$
$$p^a_{3,2}(q) = 3q^{12} + 6q^{11} + 15q^{10} + 22q^9 + 29q^8 + 28q^7 + 25q^6 + 18q^5 + 13q^4 + 8q^3 + 5q^2 + 2q^1 + 1q^0$$
$$\chi^a_{4,2} = 369$$
$$p^a_{4,2}(q) = 3q^{16} + 6q^{15} + 15q^{14} + 24q^{13} + 37q^{12} + 44q^{11} + 49q^{10} + 46q^9 + 41q^8 + 32q^7 + 25q^6 + 18q^5 + 13q^4 + 8q^3 + 5q^2 + 2q^1 + 1q^0$$
$$\chi^a_{5,2} = 671$$
$$p^a_{5,2}(q) = 3q^{20} + 6q^{19} + 15q^{18} + 24q^{17} + 39q^{16} + 52q^{15} + 65q^{14} + 70q^{13} + 73q^{12} + 68q^{11} + 61q^{10} + 50q^9 + 41q^8 + 32q^7 + 25q^6 + 18q^5 + 13q^4 + 8q^3 + 5q^2 + 2q^1 + 1q^0$$
$$\chi^a_{6,2} = 1105$$
$$p^a_{6,2}(q) = 3q^{24} + 6q^{23} + 15q^{22} + 24q^{21} + 39q^{20} + 54q^{19} + 73q^{18} + 86q^{17} + 97q^{16} + 100q^{15} + 101q^{14} + 94q^{13} + 85q^{12} + 72q^{11} + 61q^{10} + 50q^9 + 41q^8 + 32q^7 + 25q^6 + 18q^5 + 13q^4 + 8q^3 + 5q^2 + 2q^1 + 1q^0$$

EXAMPLE C.4. For $\omega \in [6]$, $n = 2$ and $\mathbf{c} = (1, 0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^{\mathbf{c}}_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$
\begin{aligned}
\chi^{\mathbf{c}}_{1,2} &= 8\\
p^{\mathbf{c}}_{1,2}(q) &= 1q^3 + 4q^2 + 2q^1 + 1q^0\\
\chi^{\mathbf{c}}_{2,2} &= 21\\
p^{\mathbf{c}}_{2,2}(q) &= 2q^5 + 7q^4 + 5q^3 + 4q^2 + 2q^1 + 1q^0\\
\chi^{\mathbf{c}}_{3,2} &= 40\\
p^{\mathbf{c}}_{3,2}(q) &= 3q^7 + 10q^6 + 8q^5 + 7q^4 + 5q^3 + 4q^2 + 2q^1 + 1q^0\\
\chi^{\mathbf{c}}_{4,2} &= 65\\
p^{\mathbf{c}}_{4,2}(q) &= 4q^9 + 13q^8 + 11q^7 + 10q^6 + 8q^5 + 7q^4 + 5q^3 + 4q^2 + 2q^1 + 1q^0\\
\chi^{\mathbf{c}}_{5,2} &= 96\\
p^{\mathbf{c}}_{5,2}(q) &= 5q^{11} + 16q^{10} + 14q^9 + 13q^8 + 11q^7 + 10q^6 + 8q^5 + 7q^4 + 5q^3 + 4q^2 + 2q^1 + 1q^0\\
\chi^{\mathbf{c}}_{6,2} &= 133\\
p^{\mathbf{c}}_{6,2}(q) &= 6q^{13} + 19q^{12} + 17q^{11} + 16q^{10} + 14q^9 + 13q^8 + 11q^7\\
&\quad + 10q^6 + 8q^5 + 7q^4 + 5q^3 + 4q^2 + 2q^1 + 1q^0
\end{aligned}
$$

EXAMPLE C.5. For $\omega \in [6]$ and $n = 2$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$
\begin{aligned}
\chi_{1,2} &= 5\\
p_{1,2}(q) &= 2q^2 + 2q^1 + 1q^0\\
\chi_{2,2} &= 9\\
p_{2,2}(q) &= 2q^4 + 2q^3 + 2q^2 + 2q^1 + 1q^0\\
\chi_{3,2} &= 13\\
p_{3,2}(q) &= 2q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + 2q^1 + 1q^0\\
\chi_{4,2} &= 17\\
p_{4,2}(q) &= 2q^8 + 2q^7 + 2q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + 2q^1 + 1q^0\\
\chi_{5,2} &= 21\\
p_{5,2}(q) &= 2q^{10} + 2q^9 + 2q^8 + 2q^7 + 2q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + 2q^1 + 1q^0\\
\chi_{6,2} &= 25\\
p_{6,2}(q) &= 2q^{12} + 2q^{11} + 2q^{10} + 2q^9 + 2q^8 + 2q^7\\
&\quad + 2q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + 2q^1 + 1q^0
\end{aligned}
$$

C.1.3. Poincaré Polynomials for n=3.

EXAMPLE C.6. For $\omega \in [4]$ and $n = 3$ the Euler characteristic and Poincaré polynomials of the approximations $\mathcal{F}l^a_\omega(\widehat{\mathfrak{gl}}_n)$ are given by

$$
\begin{aligned}
\chi^a_{1,3} &= 226\\
p^a_{1,3}(q) &= 7q^9 + 21q^8 + 39q^7 + 49q^6 + 45q^5 + 33q^4 + 19q^3 + 9q^2 + 3q^1 + 1q^0\\
\chi^a_{2,3} &= 3511\\
p^a_{2,3}(q) &= 7q^{18} + 33q^{17} + 102q^{16} + 205q^{15} + 330q^{14} + 423q^{13}\\
&\quad + 478q^{12} + 468q^{11} + 423q^{10} + 343q^9 + 261q^8 + 180q^7\\
&\quad + 118q^6 + 69q^5 + 39q^4 + 19q^3 + 9q^2 + 3q^1 + 1q^0\\
\chi^a_{3,3} &= 20620\\
p^a_{3,3}(q) &= 7q^{27} + 33q^{26} + 114q^{25} + 274q^{24} + 540q^{23} + 882q^{22} + 1258q^{21} + 1596q^{20} + 1851q^{19}\\
&\quad + 1993q^{18} + 2016q^{17} + 1932q^{16} + 1762q^{15} + 1536q^{14} + 1281q^{13}\\
&\quad + 1027q^{12} + 789q^{11} + 585q^{10} + 415q^9 + 285q^8 + 186q^7\\
&\quad + 118q^6 + 69q^5 + 39q^4 + 19q^3 + 9q^2 + 3q^1 + 1q^0\\
\chi^a_{4,3} &= 76177\\
p^a_{4,3}(q) &= 7q^{36} + 33q^{35} + 114q^{34} + 286q^{33} + 609q^{32} + 1098q^{31}\\
&\quad + 1771q^{30} + 2556q^{29} + 3405q^{28} + 4195q^{27} + 4884q^{26} + 5382q^{25}\\
&\quad + 5701q^{24} + 5796q^{23} + 5715q^{22} + 5446q^{21} + 5055q^{20} + 4551q^{19}\\
&\quad + 4003q^{18} + 3423q^{17} + 2865q^{16} + 2332q^{15} + 1860q^{14} + 1443q^{13}\\
&\quad + 1099q^{12} + 813q^{11} + 591q^{10} + 415q^9 + 285q^8 + 186q^7\\
&\quad + 118q^6 + 69q^5 + 39q^4 + 19q^3 + 9q^2 + 3q^1 + 1q^0
\end{aligned}
$$

EXAMPLE C.7. For $\omega \in [4]$, $n = 3$ and $\mathbf{c} = (1, 1, 0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^{\mathbf{c}}_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi^{\mathbf{c}}_{1,3} &= 99 \\
p^{\mathbf{c}}_{1,3}(q) &= 1q^8 + 4q^7 + 17q^6 + 25q^5 + 24q^4 + 16q^3 + 8q^2 + 3q^1 + 1q^0 \\
\chi^{\mathbf{c}}_{2,3} &= 875 \\
p^{\mathbf{c}}_{2,3}(q) &= 2q^{14} + 12q^{13} + 49q^{12} + 98q^{11} + 139q^{10} + 148q^9 + 137q^8 + 106q^7 \\
&\quad + 77q^6 + 49q^5 + 30q^4 + 16q^3 + 8q^2 + 3q^1 + 1q^0 \\
\chi^{\mathbf{c}}_{3,3} &= 3565 \\
p^{\mathbf{c}}_{3,3}(q) &= 3q^{20} + 20q^{19} + 85q^{18} + 186q^{17} + 301q^{16} + 386q^{15} + 439q^{14} + 441q^{13} \\
&\quad + 408q^{12} + 346q^{11} + 280q^{10} + 215q^9 + 159q^8 + 112q^7 \\
&\quad + 77q^6 + 49q^5 + 30q^4 + 16q^3 + 8q^2 + 3q^1 + 1q^0 \\
\chi^{\mathbf{c}}_{4,3} &= 10065 \\
p^{\mathbf{c}}_{4,3}(q) &= 4q^{26} + 28q^{25} + 121q^{24} + 278q^{23} + 477q^{22} + 664q^{21} + 827q^{20} + 932q^{19} \\
&\quad + 989q^{18} + 972q^{17} + 909q^{16} + 804q^{15} + 693q^{14} + 575q^{13} \\
&\quad + 468q^{12} + 368q^{11} + 286q^{10} + 215q^9 + 159q^8 + 112q^7 \\
&\quad + 77q^6 + 49q^5 + 30q^4 + 16q^3 + 8q^2 + 3q^1 + 1q^0
\end{aligned}$$

EXAMPLE C.8. For $\omega \in [4]$, $n = 3$ and $\mathbf{c} = (1, 0, 0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^{\mathbf{c}}_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi^{\mathbf{c}}_{1,3} &= 48 \\
p^{\mathbf{c}}_{1,2}(q) &= 1q^6 + 9q^5 + 14q^4 + 13q^3 + 7q^2 + 3q^1 + 1q^0 \\
\chi^{\mathbf{c}}_{2,3} &= 257 \\
p^{\mathbf{c}}_{2,3}(q) &= 1q^{11} + 11q^{10} + 29q^9 + 47q^8 + 50q^7 \\
&\quad + 43q^6 + 31q^5 + 21q^4 + 13q^3 + 7q^2 + 3q^1 + 1q^0 \\
\chi^{\mathbf{c}}_{3,3} &= 748 \\
p^{\mathbf{c}}_{3,3}(q) &= 1q^{16} + 11q^{15} + 31q^{14} + 63q^{13} \\
&\quad + 91q^{12} + 107q^{11} + 104q^{10} + 91q^9 + 73q^8 + 57q^7 \\
&\quad + 43q^6 + 31q^5 + 21q^4 + 13q^3 + 7q^2 + 3q^1 + 1q^0 \\
\chi^{\mathbf{c}}_{4,3} &= 1641 \\
p^{\mathbf{c}}_{4,3}(q) &= 1q^{21} + 11q^{20} + 31q^{19} \\
&\quad + 65q^{18} + 107q^{17} + 149q^{16} + 175q^{15} + 185q^{14} + 176q^{13} \\
&\quad + 157q^{12} + 133q^{11} + 111q^{10} + 91q^9 + 73q^8 + 57q^7 \\
&\quad + 43q^6 + 31q^5 + 21q^4 + 13q^3 + 7q^2 + 3q^1 + 1q^0
\end{aligned}$$

EXAMPLE C.9. For $\omega \in [6]$ and $n = 3$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l_\omega(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,3} &= 25\\
p_{1,3}(q) &= 6q^4 + 9q^3 + 6q^2 + 3q^1 + 1q^0\\
\chi_{2,3} &= 85\\
p_{2,3}(q) &= 6q^8 + 15q^7 + 18q^6 + 15q^5 + 12q^4 + 9q^3 + 6q^2 + 3q^1 + 1q^0\\
\chi_{3,3} &= 181\\
p_{3,3}(q) &= 6q^{12} + 15q^{11} + 24q^{10} + 27q^9 + 24q^8 + 21q^7\\
&\quad + 18q^6 + 15q^5 + 12q^4 + 9q^3 + 6q^2 + 3q^1 + 1q^0\\
\chi_{4,3} &= 313\\
p_{4,3}(q) &= 6q^{16} + 15q^{15} + 24q^{14} + 33q^{13}\\
&\quad + 36q^{12} + 33q^{11} + 30q^{10} + 27q^9 + 24q^8 + 21q^7\\
&\quad + 18q^6 + 15q^5 + 12q^4 + 9q^3 + 6q^2 + 3q^1 + 1q^0\\
\chi_{5,3} &= 481\\
p_{5,3}(q) &= 6q^{20} + 15q^{19} + 24q^{18} + 33q^{17} + 42q^{16} + 45q^{15} + 42q^{14} + 39q^{13}\\
&\quad + 36q^{12} + 33q^{11} + 30q^{10} + 27q^9 + 24q^8 + 21q^7\\
&\quad + 18q^6 + 15q^5 + 12q^4 + 9q^3 + 6q^2 + 3q^1 + 1q^0\\
\chi_{6,3} &= 685\\
p_{6,3}(q) &= 6q^{24} + 15q^{23} + 24q^{22} + 33q^{21} + 42q^{20} + 51q^{19}\\
&\quad + 54q^{18} + 51q^{17} + 48q^{16} + 45q^{15} + 42q^{14} + 39q^{13}\\
&\quad + 36q^{12} + 33q^{11} + 30q^{10} + 27q^9 + 24q^8 + 21q^7\\
&\quad + 18q^6 + 15q^5 + 12q^4 + 9q^3 + 6q^2 + 3q^1 + 1q^0
\end{aligned}$$

C.1.4. Poincaré Polynomials for n=4.

EXAMPLE C.10. For $\omega \in [2]$ and $n = 4$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^a_\omega(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi^a_{1,4} &= 6137\\
p^a_{1,4}(q) &= 19q^{16} + 88q^{15} + 254q^{14} + 492q^{13}\\
&\quad + 753q^{12} + 920q^{11} + 966q^{10} + 864q^9 + 689q^8 + 480q^7\\
&\quad + 304q^6 + 168q^5 + 85q^4 + 36q^3 + 14q^2 + 4q^1 + 1q^0\\
\chi^a_{2,4} &= 359313\\
p^a_{2,4}(q) &= 19q^{32} + 148q^{31}\\
&\quad + 646q^{30} + 1896q^{29} + 4343q^{28} + 8144q^{27} + 13192q^{26} + 18880q^{25}\\
&\quad + 24529q^{24} + 29260q^{23} + 32548q^{22} + 33952q^{21} + 33541q^{20} + 31456q^{19}\\
&\quad + 28206q^{18} + 24192q^{17} + 19957q^{16} + 15812q^{15} + 12088q^{14} + 8888q^{13}\\
&\quad + 6313q^{12} + 4308q^{11} + 2838q^{10} + 1788q^9 + 1085q^8 + 624q^7\\
&\quad + 344q^6 + 176q^5 + 85q^4 + 36q^3 + 14q^2 + 4q^1 + 1q^0
\end{aligned}$$

EXAMPLE C.11. For $\omega \in [3]$, $n = 4$ and $\mathbf{c} = (1,1,1,0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^{\mathbf{c}}_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi^{\mathbf{c}}_{1,4} &= 2248\\
p^{\mathbf{c}}_{1,4}(q) &= 1q^{15} + 6q^{14} + 27q^{13} + 92q^{12} + 206q^{11} + 334q^{10} + 401q^{9} + 393q^{8} + 314q^{7}\\
&\quad + 221q^{6} + 132q^{5} + 71q^{4} + 32q^{3} + 13q^{2} + 4q^{1} + 1q^{0}\\
\chi^{\mathbf{c}}_{2,4} &= 72361\\
p^{\mathbf{c}}_{2,4}(q) &= 2q^{27} + 18q^{26} + 102q^{25} + 1129q^{23} + 2407q^{22} + 4071q^{21} + 5808q^{20} + 7191q^{19}\\
&\quad + 7997q^{18} + 8111q^{17} + 7655q^{16} + 6765q^{15} + 5672q^{14} + 4511q^{13}\\
&\quad + 3432q^{12} + 2489q^{11} + 1736q^{10} + 1155q^{9} + 738q^{8} + 447q^{7}\\
&\quad + 259q^{6} + 140q^{5} + 71q^{4} + 32q^{3} + 13q^{2} + 4q^{1} + 1q^{0}\\
\chi^{\mathbf{c}}_{3,4} &= 645352\\
p^{\mathbf{c}}_{3,4}(q) &= 3q^{39} + 30q^{38} + 183q^{37}\\
&\quad + 775q^{36} + 2338q^{35} + 5486q^{34} + 10495q^{33} + 17290q^{32} + 25210q^{31}\\
&\quad + 33413q^{30} + 40822q^{29} + 46694q^{28} + 50443q^{27} + 51988q^{26} + 51383q^{25}\\
&\quad + 49035q^{24} + 45296q^{23} + 40679q^{22} + 35538q^{21} + 30296q^{20} + 25199q^{19}\\
&\quad + 20505q^{18} + 16310q^{17} + 12713q^{16} + 9690q^{15} + 7236q^{14} + 5277q^{13}\\
&\quad + 3766q^{12} + 2617q^{11} + 1774q^{10} + 1163q^{9} + 738q^{8} + 447q^{7}\\
&\quad + 259q^{6} + 140q^{5} + 71q^{4} + 32q^{3} + 13q^{2} + 4q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.12. For $\omega \in [3]$, $n = 4$ and $\mathbf{c} = (1,1,0,0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^{\mathbf{c}}_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi^{\mathbf{c}}_{1,4} &= 901\\
p^{\mathbf{c}}_{1,4}(q) &= 1q^{13} + 4q^{12} + 17q^{11} + 55q^{10} + 119q^{9} + 175q^{8} + 179q^{7}\\
&\quad + 148q^{6} + 100q^{5} + 58q^{4} + 28q^{3} + 12q^{2} + 4q^{1} + 1q^{0}\\
\chi^{\mathbf{c}}_{2,4} &= 16537\\
p^{\mathbf{c}}_{2,4}(q) &= 1q^{23} + 6q^{22} + 34q^{21} + 136q^{20} + 398q^{19}\\
&\quad + 866q^{18} + 1419q^{17} + 1867q^{16} + 2082q^{15} + 2081q^{14} + 1903q^{13}\\
&\quad + 1620q^{12} + 1289q^{11} + 970q^{10} + 692q^{9} + 471q^{8} + 304q^{7}\\
&\quad + 187q^{6} + 108q^{5} + 58q^{4} + 28q^{3} + 12q^{2} + 4q^{1} + 1q^{0}\\
\chi^{\mathbf{c}}_{3,4} &= 102805\\
p^{\mathbf{c}}_{3,4}(q) &= 1q^{33} + 6q^{32} + 36q^{31}\\
&\quad + 156q^{30} + 512q^{29} + 1307q^{28} + 2657q^{27} + 4458q^{26} + 6319q^{25}\\
&\quad + 7859q^{24} + 8864q^{23} + 9345q^{22} + 9338q^{21} + 8926q^{20} + 8191q^{19}\\
&\quad + 7252q^{18} + 6208q^{17} + 5162q^{16} + 4176q^{15} + 3297q^{14} + 2540q^{13}\\
&\quad + 1912q^{12} + 1404q^{11} + 1006q^{10} + 700q^{9} + 471q^{8} + 304q^{7}\\
&\quad + 187q^{6} + 108q^{5} + 58q^{4} + 28q^{3} + 12q^{2} + 4q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.13. For $\omega \in [3]$, $n = 4$ and $\mathbf{c} = (1,0,1,0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^{\mathbf{c}}_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi^{\mathbf{c}}_{1,4} &= 901\\
p^{\mathbf{c}}_{1,4}(q) &= 10q^{11} + 52q^{10} + 124q^{9} + 181q^{8} + 182q^{7}\\
&\qquad + 149q^{6} + 100q^{5} + 58q^{4} + 28q^{3} + 12q^{2} + 4q^{1} + 1q^{0}\\
\chi^{\mathbf{c}}_{2,4} &= 16537\\
p^{\mathbf{c}}_{2,4}(q) &= 10q^{21} + 90q^{20} + 360q^{19}\\
&\qquad + 866q^{18} + 1448q^{17} + 1899q^{16} + 2106q^{15} + 2097q^{14} + 1912q^{13}\\
&\qquad + 1624q^{12} + 1290q^{11} + 970q^{10} + 692q^{9} + 471q^{8} + 304q^{7}\\
&\qquad + 187q^{6} + 108q^{5} + 58q^{4} + 28q^{3} + 12q^{2} + 4q^{1} + 1q^{0}\\
\chi^{\mathbf{c}}_{3,4} &= 102805\\
p^{\mathbf{c}}_{3,4}(q) &= 10q^{31} + 90q^{30} + 400q^{29} + 1184q^{28} + 2592q^{27} + 4480q^{26} + 6392q^{25}\\
&\qquad + 7937q^{24} + 8930q^{23} + 9399q^{22} + 9380q^{21} + 8956q^{20} + 8210q^{19}\\
&\qquad + 7262q^{18} + 6212q^{17} + 5163q^{16} + 4176q^{15} + 3297q^{14} + 2540q^{13}\\
&\qquad + 1912q^{12} + 1404q^{11} + 1006q^{10} + 700q^{9} + 471q^{8} + 304q^{7}\\
&\qquad + 187q^{6} + 108q^{5} + 58q^{4} + 28q^{3} + 12q^{2} + 4q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.14. For $\omega \in [3]$, $n = 4$ and $\mathbf{c} = (1,0,0,0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^{\mathbf{c}}_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi^{\mathbf{c}}_{1,4} &= 392\\
p^{\mathbf{c}}_{1,4}(q) &= 1q^{10} + 16q^{9} + 51q^{8} + 81q^{7} + 87q^{6} + 70q^{5} + 46q^{4} + 24q^{3} + 11q^{2} + 4q^{1} + 1q^{0}\\
\chi^{\mathbf{c}}_{2,4} &= 4245\\
p^{\mathbf{c}}_{2,4}(q) &= 2q^{18} + 43q^{17} + 161q^{16} + 321q^{15} + 465q^{14} + 555q^{13}\\
&\qquad + 579q^{12} + 541q^{11} + 462q^{10} + 365q^{9} + 271q^{8} + 189q^{7}\\
&\qquad + 126q^{6} + 79q^{5} + 46q^{4} + 24q^{3} + 11q^{2} + 4q^{1} + 1q^{0}\\
\chi^{\mathbf{c}}_{3,4} &= 18664\\
p^{\mathbf{c}}_{3,4}(q) &= 3q^{26} + 70q^{25} + 281q^{24} + 611q^{23} + 977q^{22} + 1311q^{21} + 1575q^{20} + 1743q^{19}\\
&\qquad + 1805q^{18} + 1765q^{17} + 1644q^{16} + 1466q^{15} + 1257q^{14} + 1040q^{13}\\
&\qquad + 836q^{12} + 654q^{11} + 501q^{10} + 374q^{9} + 271q^{8} + 189q^{7}\\
&\qquad + 126q^{6} + 79q^{5} + 46q^{4} + 24q^{3} + 11q^{2} + 4q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.15. For $\omega \in [5]$ and $n = 4$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi_{1,4} &= 185\\
p_{1,4}(q) &= 6q^8 + 24q^7 + 42q^6 + 44q^5 + 34q^4 + 20q^3 + 10q^2 + 4q^1 + 1q^0\\
\chi_{2,4} &= 1233\\
p_{2,4}(q) &= 6q^{16} + 24q^{15} + 54q^{14} + 96q^{13}\\
&\quad + 138q^{12} + 164q^{11} + 170q^{10} + 156q^9 + 130q^8 + 100q^7\\
&\quad + 74q^6 + 52q^5 + 34q^4 + 20q^3 + 10q^2 + 4q^1 + 1q^0\\
\chi_{3,4} &= 3913\\
p_{3,4}(q) &= 6q^{24} + 24q^{23} + 54q^{22} + 96q^{21} + 150q^{20} + 216q^{19}\\
&\quad + 282q^{18} + 332q^{17} + 362q^{16} + 372q^{15} + 362q^{14} + 332q^{13}\\
&\quad + 290q^{12} + 244q^{11} + 202q^{10} + 164q^9 + 130q^8 + 100q^7\\
&\quad + 74q^6 + 52q^5 + 34q^4 + 20q^3 + 10q^2 + 4q^1 + 1q^0\\
\chi_{4,4} &= 8993\\
p_{4,4}(q) &= 6q^{32} + 24q^{31} + 54q^{30} + 96q^{29} + 150q^{28} + 216q^{27} + 294q^{26} + 384q^{25}\\
&\quad + 474q^{24} + 548q^{23} + 602q^{22} + 636q^{21} + 650q^{20} + 644q^{19}\\
&\quad + 618q^{18} + 572q^{17} + 514q^{16} + 452q^{15} + 394q^{14} + 340q^{13}\\
&\quad + 290q^{12} + 244q^{11} + 202q^{10} + 164q^9 + 130q^8 + 100q^7\\
&\quad + 74q^6 + 52q^5 + 34q^4 + 20q^3 + 10q^2 + 4q^1 + 1q^0\\
\chi_{5,4} &= 17241\\
p_{5,4}(q) &= 6q^{40} + 24q^{39} + 54q^{38} + 96q^{37}\\
&\quad + 150q^{36} + 216q^{35} + 294q^{34} + 384q^{33} + 486q^{32} + 600q^{31}\\
&\quad + 714q^{30} + 812q^{29} + 890q^{28} + 948q^{27} + 986q^{26} + 1004q^{25}\\
&\quad + 1002q^{24} + 980q^{23} + 938q^{22} + 876q^{21} + 802q^{20} + 724q^{19}\\
&\quad + 650q^{18} + 580q^{17} + 514q^{16} + 452q^{15} + 394q^{14} + 340q^{13}\\
&\quad + 290q^{12} + 244q^{11} + 202q^{10} + 164q^9 + 130q^8 + 100q^7\\
&\quad + 74q^6 + 52q^5 + 34q^4 + 20q^3 + 10q^2 + 4q^1 + 1q^0
\end{aligned}$$

C.1.5. Poincaré Polynomials for n=5.

EXAMPLE C.16. For $\omega = 1$ and $n = 5$ the Euler characteristic and Poincaré polynomial of the approximation $\mathcal{F}l^a_\omega\big(\widehat{\mathfrak{gl}}_n\big)$ is given by

$$\begin{aligned}
\chi^a_{1,5} &= 265266\\
p^a_{1,5}(q) &= 51q^{25} + 355q^{24} + 1390q^{23} + 3780q^{22} + 7985q^{21} + 13841q^{20} + 20480q^{19}\\
&\quad + 26530q^{18} + 30675q^{17} + 32095q^{16} + 30716q^{15} + 27110q^{14} + 22195q^{13}\\
&\quad + 16935q^{12} + 12070q^{11} + 8056q^{10} + 5030q^9 + 2940q^8 + 1600q^7\\
&\quad + 810q^6 + 376q^5 + 160q^4 + 60q^3 + 20q^2 + 5q^1 + 1q^0
\end{aligned}$$

EXAMPLE C.17. For $\omega = 1$, $n = 5$ and $\mathbf{c} = (1,1,1,1,0)$ the Euler characteristic and Poincaré polynomial of the approximation $\mathcal{F}l_\omega^{\mathbf{c}}(\widehat{\mathfrak{gl}}_n)$ is given by

$$\begin{aligned}
\chi_{1,5}^{\mathbf{c}} &= 82865\\
p_{1,5}^{\mathbf{c}}(q) &= 1q^{24} + 8q^{23} + 46q^{22} + 186q^{21} + 634q^{20} + 1677q^{19}\\
&\quad + 3557q^{18} + 6079q^{17} + 8667q^{16} + 10548q^{15} + 11244q^{14} + 10674q^{13}\\
&\quad + 9166q^{12} + 7184q^{11} + 5184q^{10} + 3458q^{9} + 2140q^{8} + 1226q^{7}\\
&\quad + 650q^{6} + 316q^{5} + 140q^{4} + 55q^{3} + 19q^{2} + 5q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.18. For $\omega = 1$, $n = 5$ and $\mathbf{c} = (1,1,1,0,0)$ the Euler characteristic and Poincaré polynomial of the approximation $\mathcal{F}l_\omega^{\mathbf{c}}(\widehat{\mathfrak{gl}}_n)$ is given by

$$\begin{aligned}
\chi_{1,5}^{\mathbf{c}} &= 28016\\
p_{1,2}^{\mathbf{c}}(q) &= 1q^{22} + 5q^{21} + 20q^{20} + 67q^{19}\\
&\quad + 196q^{18} + 527q^{17} + 1180q^{16} + 2179q^{15} + 3241q^{14} + 3985q^{13}\\
&\quad + 4158q^{12} + 3785q^{11} + 3062q^{10} + 2239q^{9} + 1493q^{8} + 912q^{7}\\
&\quad + 510q^{6} + 261q^{5} + 121q^{4} + 50q^{3} + 18q^{2} + 5q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.19. For $\omega \in [6]$, $n = 2$ and $\mathbf{c} = (1,1,0,1,0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l_\omega^{\mathbf{c}}(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,5}^{\mathbf{c}} &= 28016\\
p_{1,5}^{\mathbf{c}}(q) &= 3q^{20} + 21q^{19} + 106q^{18} + 408q^{17} + 1097q^{16} + 2189q^{15} + 3326q^{14} + 4089q^{13}\\
&\quad + 4240q^{12} + 3833q^{11} + 3084q^{10} + 2247q^{9} + 1495q^{8} + 912q^{7}\\
&\quad + 510q^{6} + 261q^{5} + 121q^{4} + 50q^{3} + 18q^{2} + 5q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.20. For $\omega \in [6]$, $n = 2$ and $\mathbf{c} = (1,0,1,0,0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l_\omega^{\mathbf{c}}(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,5}^{\mathbf{c}} &= 10221\\
p_{1,5}^{\mathbf{c}}(q) &= 4q^{17} + 28q^{16} + 151q^{15} + 498q^{14} + 1014q^{13}\\
&\quad + 1480q^{12} + 1693q^{11} + 1615q^{10} + 1335q^{9} + 982q^{8} + 650q^{7}\\
&\quad + 389q^{6} + 211q^{5} + 103q^{4} + 45q^{3} + 17q^{2} + 5q^{1} + 1q^{0}\\
\chi_{2,5}^{\mathbf{c}} &= 386777\\
p_{2,5}^{\mathbf{c}}(q) &= 10q^{31} + 97q^{30} + 675q^{29} + 2721q^{28} + 7146q^{27} + 13845q^{26} + 21687q^{25}\\
&\quad + 29116q^{24} + 34777q^{23} + 37958q^{22} + 38564q^{21} + 36969q^{20} + 33746q^{19}\\
&\quad + 29521q^{18} + 24842q^{17} + 20169q^{16} + 15827q^{15} + 12028q^{14} + 8859q^{13}\\
&\quad + 6329q^{12} + 4381q^{11} + 2937q^{10} + 1901q^{9} + 1185q^{8} + 706q^{7}\\
&\quad + 399q^{6} + 211q^{5} + 103q^{4} + 45q^{3} + 17q^{2} + 5q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.21. For $\omega \in [2]$, $n = 5$ and $\mathbf{c} = (1,1,0,0,0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^{\mathbf{c}}_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi^{\mathbf{c}}_{1,5} &= 10221\\
p^{\mathbf{c}}_{1,5}(q) &= 1q^{19} + 5q^{18} + 23q^{17} + 78q^{16} + 219q^{15} + 533q^{14} + 991q^{13}\\
&\quad + 1425q^{12} + 1643q^{11} + 1585q^{10} + 1321q^{9} + 977q^{8} + 649q^{7}\\
&\quad + 389q^{6} + 211q^{5} + 103q^{4} + 45q^{3} + 17q^{2} + 5q^{1} + 1q^{0}\\
\chi^{\mathbf{c}}_{2,5} &= 386777\\
p^{\mathbf{c}}_{2,5}(q) &= 2q^{33} + 16q^{32} + 93q^{31}\\
&\quad + 384q^{30} + 1279q^{29} + 3505q^{28} + 7782q^{27} + 14099q^{26} + 21536q^{25}\\
&\quad + 28695q^{24} + 34263q^{23} + 37478q^{22} + 38179q^{21} + 36689q^{20} + 33556q^{19}\\
&\quad + 29401q^{18} + 24773q^{17} + 20134q^{16} + 15812q^{15} + 12023q^{14} + 8858q^{13}\\
&\quad + 6329q^{12} + 4381q^{11} + 2937q^{10} + 1901q^{9} + 1185q^{8} + 706q^{7}\\
&\quad + 399q^{6} + 211q^{5} + 103q^{4} + 45q^{3} + 17q^{2} + 5q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.22. For $\omega \in [2]$, $n = 5$ and $\mathbf{c} = (1,0,0,0,0)$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l^{\mathbf{c}}_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi^{\mathbf{c}}_{1,5} &= 4020\\
p^{\mathbf{c}}_{1,5}(q) &= 1q^{15} + 25q^{14} + 129q^{13}\\
&\quad + 328q^{12} + 550q^{11} + 685q^{10} + 687q^{9} + 584q^{8} + 433q^{7}\\
&\quad + 284q^{6} + 166q^{5} + 86q^{4} + 40q^{3} + 16q^{2} + 5q^{1} + 1q^{0}\\
\chi^{\mathbf{c}}_{2,5} &= 88361\\
p^{\mathbf{c}}_{2,5}(q) &= 1q^{28} + 27q^{27} + 204q^{26} + 790q^{25}\\
&\quad + 1967q^{24} + 3654q^{23} + 5520q^{22} + 7181q^{21} + 8369q^{20} + 8962q^{19}\\
&\quad + 8971q^{18} + 8479q^{17} + 7622q^{16} + 6545q^{15} + 5391q^{14} + 4269q^{13}\\
&\quad + 3258q^{12} + 2399q^{11} + 1706q^{10} + 1171q^{9} + 775q^{8} + 491q^{7}\\
&\quad + 295q^{6} + 166q^{5} + 86q^{4} + 40q^{3} + 16q^{2} + 5q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.23. For $\omega \in [2]$ and $n = 5$ the Euler characteristics and Poincaré polynomials of the approximations $\mathcal{F}l_{\omega}\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi_{1,5} &= 1701\\
p_{1,5}(q) &= 30q^{12} + 120q^{11} + 230q^{10} + 300q^{9} + 310q^{8} + 265q^{7}\\
&\quad + 195q^{6} + 125q^{5} + 70q^{4} + 35q^{3} + 15q^{2} + 5q^{1} + 1q^{0}\\
\chi_{2,5} &= 22421\\
p_{2,5}(q) &= 30q^{24} + 180q^{23} + 490q^{22} + 900q^{21} + 1330q^{20} + 1720q^{19}\\
&\quad + 2030q^{18} + 2220q^{17} + 2270q^{16} + 2185q^{15} + 1995q^{14} + 1735q^{13}\\
&\quad + 1440q^{12} + 1145q^{11} + 875q^{10} + 645q^{9} + 460q^{8} + 315q^{7}\\
&\quad + 205q^{6} + 125q^{5} + 70q^{4} + 35q^{3} + 15q^{2} + 5q^{1} + 1q^{0}
\end{aligned}$$

C.1.6. Poincaré Polynomials for n=6.

EXAMPLE C.24. For $\omega = 1$ and $n = 6$ the Euler characteristic and Poincaré polynomial of the approximation $\mathcal{F}l_\omega(\widehat{\mathfrak{gl}}_n)$ is given by

$$\begin{aligned} \chi_{1,6} &= 19045 \\ p_{1,6}(q) &= 20q^{18} + 180q^{17} + 630q^{16} + 1340q^{15} + 2085q^{14} + 2610q^{13} \\ &\quad + 2780q^{12} + 2610q^{11} + 2205q^{10} + 1694q^{9} + 1194q^{8} + 774q^{7} \\ &\quad + 461q^{6} + 252q^{5} + 126q^{4} + 56q^{3} + 21q^{2} + 6q^{1} + 1q^{0} \end{aligned}$$

C.2. For Approximations of Partial-Degenerate Affine Grassmannians

The computations of the Poincaré polynomials for the finite approximations of the partial degenerations of the affine Grassmannian show that in this setting the Poincaré polynomials only depend on the number of projections, i.e. the codimension of the map $f : V \to V$ and not the relative positions of the projections. Hence we only give the Poincaré polynomial of one representative for these isomorphism classes of degenerations in the following list of examples. Polynomials where computed using the formula for the Poincaré polynomials of the loop quiver as introduced in Chapter 5 and implemented as in Appendix B.2.

C.2.1. Poincaré Polynomials for n=1.

EXAMPLE C.25. For $N \in \mathbb{N}$ and $n = 1$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}^a_N(\widehat{\mathfrak{gl}}_n)$ are given by

$$\chi^a_{N,1} = N + 1$$
$$p^a_{N,1}(q) = \sum_{k=0}^{N} q^k$$

EXAMPLE C.26. For $N \in \mathbb{N}$ and $n = 1$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N(\widehat{\mathfrak{gl}}_n)$ are given by

$$\chi_{N,1} = 1$$
$$p_{N,1}(q) = q^0$$

C.2.2. Poincaré Polynomials for n=2.

EXAMPLE C.27. For $N \in [5]$ and $n = 2$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}^a_N(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi^a_{1,2} &= 6 \\
p^a_{1,2}(q) &= 1q^4 + 1q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi^a_{2,2} &= 19 \\
p^a_{2,2}(q) &= 1q^8 + 1q^7 + 3q^6 + 3q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi^a_{3,2} &= 44 \\
p^a_{3,2}(q) &= 1q^{12} + 1q^{11} + 3q^{10} + 4q^9 + 6q^8 + 6q^7 \\
&\quad + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi^a_{4,2} &= 85 \\
p^a_{4,2}(q) &= 1q^{16} + 1q^{15} + 3q^{14} + 4q^{13} \\
&\quad + 7q^{12} + 8q^{11} + 10q^{10} + 10q^9 + 10q^8 + 8q^7 \\
&\quad + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi^a_{5,2} &= 146 \\
p^a_{5,2}(q) &= 1q^{20} + 1q^{19} + 3q^{18} + 4q^{17} + 7q^{16} + 9q^{15} + 12q^{14} + 13q^{13} \\
&\quad + 15q^{12} + 14q^{11} + 14q^{10} + 12q^9 + 10q^8 + 8q^7 \\
&\quad + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}$$

EXAMPLE C.28. For $N \in [5]$, $n = 2$ and $k = 1$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,2}^k &= 4 \\
p_{1,2}^k(q) &= 2q^2 + 1q^1 + 1q^0 \\
\chi_{2,2}^k &= 9 \\
p_{2,2}^k(q) &= 3q^4 + 2q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi_{3,2}^k &= 16 \\
p_{3,2}^k(q) &= 4q^6 + 3q^5 + 3q^4 + 2q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi_{4,2}^k &= 25 \\
p_{4,2}^k(q) &= 5q^8 + 4q^7 + 4q^6 + 3q^5 + 3q^4 + 2q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi_{5,2}^k &= 36 \\
p_{5,2}^k(q) &= 6q^{10} + 5q^9 + 5q^8 + 4q^7 + 4q^6 + 3q^5 + 3q^4 + 2q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}$$

EXAMPLE C.29. For $N \in [5]$ and $n = 2$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,2} &= 3 \\
p_{1,2}(q) &= 1q^2 + 1q^1 + 1q^0 \\
\chi_{2,2} &= 5 \\
p_{2,2}(q) &= 1q^4 + 1q^3 + 1q^2 + 1q^1 + 1q^0 \\
\chi_{3,2} &= 7 \\
p_{3,2}(q) &= 1q^6 + 1q^5 + 1q^4 + 1q^3 + 1q^2 + 1q^1 + 1q^0 \\
\chi_{4,2} &= 9 \\
p_{4,2}(q) &= 1q^8 + 1q^7 + 1q^6 + 1q^5 + 1q^4 + 1q^3 + 1q^2 + 1q^1 + 1q^0 \\
\chi_{5,2} &= 11 \\
p_{5,2}(q) &= 1q^{10} + 1q^9 + 1q^8 + 1q^7 + 1q^6 + 1q^5 + 1q^4 + 1q^3 + 1q^2 + 1q^1 + 1q^0
\end{aligned}$$

C.2.3. Poincaré Polynomials for n=3.

EXAMPLE C.30. For $N \in [5]$ and $n = 3$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^a(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi^a_{1,3} &= 20\\
p^a_{1,3}(q) &= 1q^9 + 1q^8 + 2q^7 + 3q^6 + 3q^5 + 3q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^a_{2,3} &= 141\\
p^a_{2,3}(q) &= 1q^{18} + 1q^{17} + 3q^{16} + 5q^{15} + 8q^{14} + 10q^{13}\\
&\quad + 14q^{12} + 14q^{11} + 16q^{10} + 15q^9 + 14q^8 + 11q^7\\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^a_{3,3} &= 58\\
p^a_{3,3}(q) &= 1q^{27} + 1q^{26} + 3q^{25} + 6q^{24} + 10q^{23} + 15q^{22} + 23q^{21} + 29q^{20} + 36q^{19}\\
&\quad + 42q^{18} + 46q^{17} + 48q^{16} + 48q^{15} + 46q^{14} + 43q^{13}\\
&\quad + 39q^{12} + 33q^{11} + 28q^{10} + 23q^9 + 18q^8 + 13q^7\\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^a_{4,3} &= 1751\\
p^a_{4,3}(q) &= 1q^{36} + 1q^{35} + 3q^{34} + 6q^{33} + 11q^{32} + 17q^{31}\\
&\quad + 28q^{30} + 38q^{29} + 53q^{28} + 66q^{27} + 81q^{26} + 92q^{25}\\
&\quad + 105q^{24} + 110q^{23} + 116q^{22} + 116q^{21} + 116q^{20} + 110q^{19}\\
&\quad + 105q^{18} + 95q^{17} + 87q^{16} + 76q^{15} + 66q^{14} + 55q^{13}\\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^9 + 18q^8 + 13q^7\\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^a_{5,3} &= 4332\\
p^a_{5,3}(q) &= 1q^{45} + 1q^{44} + 3q^{43} + 6q^{42} + 11q^{41} + 18q^{40} + 30q^{39} + 43q^{38} + 62q^{37}\\
&\quad + 83q^{36} + 107q^{35} + 131q^{34} + 157q^{33} + 179q^{32} + 200q^{31}\\
&\quad + 217q^{30} + 229q^{29} + 237q^{28} + 241q^{27} + 240q^{26} + 235q^{25}\\
&\quad + 227q^{24} + 215q^{23} + 201q^{22} + 185q^{21} + 168q^{20} + 150q^{19}\\
&\quad + 133q^{18} + 115q^{17} + 99q^{16} + 84q^{15} + 70q^{14} + 57q^{13}\\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^9 + 18q^8 + 13q^7\\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}$$

EXAMPLE C.31. For $N \in [5]$, $n = 3$ and $k = 2$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n)$ are given by

$$
\begin{aligned}
\chi_{1,3}^k &= 14 \\
p_{1,3}^k(q) &= 2q^6 + 2q^5 + 3q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi_{2,3}^k &= 71 \\
p_{2,3}^k(q) &= 3q^{12} + 4q^{11} + 8q^{10} + 9q^9 + 11q^8 + 9q^7 \\
&\quad + 9q^6 + 6q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi_{3,3}^k &= 226 \\
p_{3,3}^k(q) &= 4q^{18} + 6q^{17} + 13q^{16} + 17q^{15} + 22q^{14} + 23q^{13} \\
&\quad + 26q^{12} + 23q^{11} + 21q^{10} + 18q^9 + 15q^8 + 11q^7 \\
&\quad + 9q^6 + 6q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi_{4,3}^k &= 555 \\
p_{4,3}^k(q) &= 5q^{24} + 8q^{23} + 18q^{22} + 25q^{21} + 35q^{20} + 39q^{19} \\
&\quad + 47q^{18} + 47q^{17} + 50q^{16} + 46q^{15} + 43q^{14} + 37q^{13} \\
&\quad + 34q^{12} + 27q^{11} + 23q^{10} + 18q^9 + 15q^8 + 11q^7 \\
&\quad + 9q^6 + 6q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi_{5,3}^k &= 1156 \\
p_{5,3}^k(q) &= 6q^{30} + 10q^{29} + 23q^{28} + 33q^{27} + 48q^{26} + 57q^{25} \\
&\quad + 70q^{24} + 75q^{23} + 83q^{22} + 84q^{21} + 86q^{20} + 80q^{19} \\
&\quad + 76q^{18} + 68q^{17} + 62q^{16} + 54q^{15} + 47q^{14} + 39q^{13} \\
&\quad + 34q^{12} + 27q^{11} + 23q^{10} + 18q^9 + 15q^8 + 11q^7 \\
&\quad + 9q^6 + 6q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}
$$

EXAMPLE C.32. For $N \in [5]$, $n = 3$ and $k = 1$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,3}^k &= 10\\
p_{1,3}^k(q) &= 1q^5 + 2q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{2,3}^k &= 37\\
p_{2,3}^k(q) &= 1q^{10} + 2q^9 + 5q^8 + 6q^7 + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{3,3}^k &= 92\\
p_{3,3}^k(q) &= 1q^{15} + 2q^{14} + 5q^{13} + 8q^{12} + 11q^{11} + 12q^{10} + 12q^9 + 10q^8 + 8q^7\\
&\quad + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{4,3}^k &= 185\\
p_{4,3}^k(q) &= 1q^{20} + 2q^{19} + 5q^{18} + 8q^{17} + 13q^{16} + 16q^{15} + 19q^{14} + 19q^{13}\\
&\quad + 19q^{12} + 16q^{11} + 14q^{10} + 12q^9 + 10q^8 + 8q^7\\
&\quad + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{5,3}^k &= 326\\
p_{5,3}^k(q) &= 1q^{25} + 2q^{24} + 5q^{23} + 8q^{22} + 13q^{21} + 18q^{20} + 23q^{19}\\
&\quad + 26q^{18} + 28q^{17} + 28q^{16} + 27q^{15} + 24q^{14} + 21q^{13}\\
&\quad + 19q^{12} + 16q^{11} + 14q^{10} + 12q^9 + 10q^8 + 8q^7\\
&\quad + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}$$

EXAMPLE C.33. For $N \in [5]$ and $n = 3$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,3} &= 7\\
p_{1,3}(q) &= 1q^4 + 2q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{2,3} &= 19\\
p_{2,3}(q) &= 1q^8 + 2q^7 + 4q^6 + 3q^5 + 3q^4 + 2q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{3,3} &= 37\\
p_{3,3}(q) &= 1q^{12} + 2q^{11} + 4q^{10} + 5q^9 + 5q^8 + 4q^7\\
&\quad + 4q^6 + 3q^5 + 3q^4 + 2q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{4,3} &= 61\\
p_{4,3}(q) &= 1q^{16} + 2q^{15} + 4q^{14} + 5q^{13} + 7q^{12} + 6q^{11} + 6q^{10} + 5q^9 + 5q^8 + 4q^7\\
&\quad + 4q^6 + 3q^5 + 3q^4 + 2q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{5,3} &= 91\\
p_{5,3}(q) &= 1q^{20} + 2q^{19} + 4q^{18} + 5q^{17} + 7q^{16} + 8q^{15} + 8q^{14} + 7q^{13}\\
&\quad + 7q^{12} + 6q^{11} + 6q^{10} + 5q^9 + 5q^8 + 4q^7\\
&\quad + 4q^6 + 3q^5 + 3q^4 + 2q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}$$

C.2.4. Poincaré Polynomials for n=4.

EXAMPLE C.34. For $N \in [5]$ and $n = 4$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}^a_N(\widehat{\mathfrak{gl}}_n)$ are given by

$$
\begin{aligned}
\chi^a_{1,4} &= 70\\
p^a_{1,4}(q) &= 1q^{16} + 1q^{15} + 2q^{14} + 3q^{13} + 5q^{12} + 5q^{11} + 7q^{10} + 7q^9 + 8q^8 + 7q^7\\
&\quad + 7q^6 + 5q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^a_{2,4} &= 1107\\
p^a_{2,4}(q) &= 1q^{32} + 1q^{31} + 3q^{30} + 5q^{29} + 10q^{28} + 14q^{27} + 22q^{26} + 29q^{25}\\
&\quad + 40q^{24} + 48q^{23} + 59q^{22} + 66q^{21} + 75q^{20} + 78q^{19}\\
&\quad + 82q^{18} + 80q^{17} + 79q^{16} + 72q^{15} + 67q^{14} + 58q^{13}\\
&\quad + 51q^{12} + 41q^{11} + 34q^{10} + 26q^9 + 21q^8 + 15q^7\\
&\quad + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^a_{3,4} &= 8092\\
p^a_{3,4}(q) &= 1q^{48} + 1q^{47} + 3q^{46} + 6q^{45} + 12q^{44} + 19q^{43}\\
&\quad + 33q^{42} + 48q^{41} + 72q^{40} + 98q^{39} + 132q^{38} + 167q^{37}\\
&\quad + 211q^{36} + 249q^{35} + 294q^{34} + 332q^{33} + 371q^{32} + 398q^{31}\\
&\quad + 426q^{30} + 438q^{29} + 449q^{28} + 446q^{27} + 439q^{26} + 420q^{25}\\
&\quad + 402q^{24} + 371q^{23} + 343q^{22} + 308q^{21} + 276q^{20} + 240q^{19}\\
&\quad + 210q^{18} + 177q^{17} + 150q^{16} + 123q^{15} + 101q^{14} + 80q^{13}\\
&\quad + 65q^{12} + 49q^{11} + 38q^{10} + 28q^9 + 21q^8 + 15q^7\\
&\quad + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^a_{4,4} &= 38165\\
p^a_{4,4}(q) &= 1q^{64} + 1q^{63} + 3q^{62} + 6q^{61}\\
&\quad + 13q^{60} + 21q^{59} + 38q^{58} + 59q^{57} + 93q^{56} + 134q^{55}\\
&\quad + 192q^{54} + 258q^{53} + 346q^{52} + 439q^{51} + 551q^{50} + 667q^{49}\\
&\quad + 798q^{48} + 923q^{47} + 1059q^{46} + 1181q^{45} + 1304q^{44} + 1408q^{43}\\
&\quad + 1507q^{42} + 1578q^{41} + 1642q^{40} + 1674q^{39} + 1695q^{38} + 1688q^{37}\\
&\quad + 1670q^{36} + 1623q^{35} + 1571q^{34} + 1496q^{33} + 1418q^{32} + 1324q^{31}\\
&\quad + 1231q^{30} + 1126q^{29} + 1029q^{28} + 925q^{27} + 829q^{26} + 732q^{25}\\
&\quad + 645q^{24} + 558q^{23} + 484q^{22} + 412q^{21} + 350q^{20} + 292q^{19}\\
&\quad + 244q^{18} + 199q^{17} + 164q^{16} + 131q^{15} + 105q^{14} + 82q^{13}\\
&\quad + 65q^{12} + 49q^{11} + 38q^{10} + 28q^9 + 21q^8 + 15q^7\\
&\quad + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}
$$

$$
\begin{aligned}
\chi^a_{5,4} &= 135954 \\
p^a_{5,4}(q) &= 1q^{80} + 1q^{79} + 3q^{78} + 6q^{77} + 13q^{76} + 22q^{75} + 40q^{74} + 64q^{73} \\
&\quad + 104q^{72} + 155q^{71} + 230q^{70} + 322q^{69} + 447q^{68} + 592q^{67} \\
&\quad + 775q^{66} + 980q^{65} + 1223q^{64} + 1482q^{63} + 1776q^{62} + 2077q^{61} \\
&\quad + 2404q^{60} + 2724q^{59} + 3058q^{58} + 3372q^{57} + 3687q^{56} + 3968q^{55} \\
&\quad + 4237q^{54} + 4461q^{53} + 4664q^{52} + 4813q^{51} + 4936q^{50} + 5001q^{49} \\
&\quad + 5038q^{48} + 5018q^{47} + 4972q^{46} + 4875q^{45} + 4757q^{44} + 4595q^{43} \\
&\quad + 4419q^{42} + 4209q^{41} + 3994q^{40} + 3753q^{39} + 3515q^{38} + 3261q^{37} \\
&\quad + 3016q^{36} + 2763q^{35} + 2524q^{34} + 2284q^{33} + 2062q^{32} + 1843q^{31} \\
&\quad + 1644q^{30} + 1451q^{29} + 1279q^{28} + 1115q^{27} + 971q^{26} + 836q^{25} \\
&\quad + 719q^{24} + 610q^{23} + 518q^{22} + 434q^{21} + 364q^{20} + 300q^{19} \\
&\quad + 248q^{18} + 201q^{17} + 164q^{16} + 131q^{15} + 105q^{14} + 82q^{13} \\
&\quad + 65q^{12} + 49q^{11} + 38q^{10} + 28q^{9} + 21q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

EXAMPLE C.35. For $N \in [5]$, $n = 4$ and $k = 3$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}^k_N\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\chi^k_{1,4} = 50$$

$$p^k_{1,4}(q) = 2q^{12} + 2q^{11} + 4q^{10} + 5q^9 + 7q^8 + 6q^7 + 7q^6 + 5q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0$$

$$\chi^k_{2,4} = 573$$

$$\begin{aligned} p^k_{2,4}(q) = {} & 3q^{24} + 4q^{23} + 10q^{22} + 15q^{21} + 25q^{20} + 31q^{19} \\ & + 41q^{18} + 45q^{17} + 52q^{16} + 51q^{15} + 51q^{14} + 46q^{13} \\ & + 44q^{12} + 36q^{11} + 31q^{10} + 24q^9 + 20q^8 + 14q^7 \\ & + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \end{aligned}$$

$$\chi^k_{3,4} = 3256$$

$$\begin{aligned} p^k_{3,4}(q) = {} & 4q^{36} + 6q^{35} + 16q^{34} + 27q^{33} + 47q^{32} + 66q^{31} \\ & + 97q^{30} + 120q^{29} + 152q^{28} + 174q^{27} + 198q^{26} + 209q^{25} \\ & + 222q^{24} + 218q^{23} + 218q^{22} + 206q^{21} + 195q^{20} + 175q^{19} \\ & + 161q^{18} + 139q^{17} + 122q^{16} + 102q^{15} + 86q^{14} + 69q^{13} \\ & + 58q^{12} + 44q^{11} + 35q^{10} + 26q^9 + 20q^8 + 14q^7 \\ & + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \end{aligned}$$

$$\chi^k_{4,4} = 12529$$

$$\begin{aligned} p^k_{4,4}(q) = {} & 5q^{48} + 8q^{47} + 22q^{46} + 39q^{45} + 71q^{44} + 105q^{43} \\ & + 161q^{42} + 214q^{41} + 287q^{40} + 352q^{39} + 428q^{38} + 489q^{37} \\ & + 558q^{36} + 601q^{35} + 645q^{34} + 665q^{33} + 683q^{32} + 676q^{31} \\ & + 670q^{30} + 641q^{29} + 617q^{28} + 576q^{27} + 538q^{26} + 489q^{25} \\ & + 448q^{24} + 396q^{23} + 353q^{22} + 307q^{21} + 268q^{20} + 227q^{19} \\ & + 195q^{18} + 161q^{17} + 136q^{16} + 110q^{15} + 90q^{14} + 71q^{13} \\ & + 58q^{12} + 44q^{11} + 35q^{10} + 26q^9 + 20q^8 + 14q^7 \\ & + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \end{aligned}$$

$$\chi^k_{5,4} = 37654$$

$$\begin{aligned} p^k_{5,4}(q) = {} & 6q^{60} + 10q^{59} + 28q^{58} + 51q^{57} + 95q^{56} + 146q^{55} \\ & + 229q^{54} + 316q^{53} + 438q^{52} + 560q^{51} + 710q^{50} + 850q^{49} \\ & + 1011q^{48} + 1147q^{47} + 1294q^{46} + 1408q^{45} + 1520q^{44} + 1593q^{43} \\ & + 1663q^{42} + 1689q^{41} + 1711q^{40} + 1694q^{39} + 1674q^{38} + 1621q^{37} \\ & + 1570q^{36} + 1490q^{35} + 1416q^{34} + 1322q^{33} + 1233q^{32} + 1130q^{31} \\ & + 1040q^{30} + 937q^{29} + 847q^{28} + 753q^{27} + 671q^{26} + 588q^{25} \\ & + 518q^{24} + 446q^{23} + 387q^{22} + 329q^{21} + 282q^{20} + 235q^{19} \\ & + 199q^{18} + 163q^{17} + 136q^{16} + 110q^{15} + 90q^{14} + 71q^{13} \\ & + 58q^{12} + 44q^{11} + 35q^{10} + 26q^9 + 20q^8 + 14q^7 \\ & + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \end{aligned}$$

EXAMPLE C.36. For $N \in [5]$, $n = 4$ and $k = 2$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi_{1,4}^k &= 36\\
p_{1,4}^k(q) &= 1q^{10} + 2q^9 + 5q^8 + 5q^7\\
&\quad + 6q^6 + 5q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{2,4}^k &= 301\\
p_{2,4}^k(q) &= 1q^{20} + 2q^{19} + 7q^{18} + 13q^{17} + 22q^{16} + 26q^{15} + 31q^{14} + 31q^{13}\\
&\quad + 32q^{12} + 29q^{11} + 26q^{10} + 21q^9 + 18q^8 + 13q^7\\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{3,4}^k &= 1336\\
p_{3,4}^k(q) &= 1q^{30} + 2q^{29} + 7q^{28} + 15q^{27} + 30q^{26} + 47q^{25}\\
&\quad + 69q^{24} + 82q^{23} + 95q^{22} + 101q^{21} + 106q^{20} + 104q^{19}\\
&\quad + 102q^{18} + 94q^{17} + 87q^{16} + 76q^{15} + 66q^{14} + 55q^{13}\\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^9 + 18q^8 + 13q^7\\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{4,4}^k &= 4209\\
p_{4,4}^k(q) &= 1q^{40} + 2q^{39} + 7q^{38} + 15q^{37}\\
&\quad + 32q^{36} + 55q^{35} + 90q^{34} + 127q^{33} + 169q^{32} + 199q^{31}\\
&\quad + 227q^{30} + 244q^{29} + 259q^{28} + 263q^{27} + 265q^{26} + 258q^{25}\\
&\quad + 250q^{24} + 234q^{23} + 218q^{22} + 197q^{21} + 178q^{20} + 156q^{19}\\
&\quad + 137q^{18} + 117q^{17} + 101q^{16} + 84q^{15} + 70q^{14} + 57q^{13}\\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^9 + 18q^8 + 13q^7\\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{5,4}^k &= 10700\\
p_{5,4}^k(q) &= 1q^{50} + 2q^{49} + 7q^{48} + 15q^{47} + 32q^{46} + 57q^{45} + 98q^{44} + 148q^{43}\\
&\quad + 214q^{42} + 282q^{41} + 354q^{40} + 410q^{39} + 462q^{38} + 498q^{37}\\
&\quad + 530q^{36} + 547q^{35} + 559q^{34} + 558q^{33} + 554q^{32} + 537q^{31}\\
&\quad + 518q^{30} + 490q^{29} + 461q^{28} + 425q^{27} + 391q^{26} + 353q^{25}\\
&\quad + 319q^{24} + 283q^{23} + 251q^{22} + 219q^{21} + 192q^{20} + 164q^{19}\\
&\quad + 141q^{18} + 119q^{17} + 101q^{16} + 84q^{15} + 70q^{14} + 57q^{13}\\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^9 + 18q^8 + 13q^7\\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}$$

EXAMPLE C.37. For $N \in [5]$, $n = 4$ and $k = 1$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}^k_N\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$
\begin{aligned}
\chi^k_{1,4} &= 26\\
p^k_{1,4}(q) &= 2q^8 + 3q^7 + 5q^6 + 4q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^k_{2,4} &= 159\\
p^k_{2,4}(q) &= 3q^{16} + 6q^{15} + 12q^{14} + 14q^{13}\\
&\qquad + 18q^{12} + 18q^{11} + 19q^{10} + 16q^9 + 15q^8 + 11q^7\\
&\qquad + 9q^6 + 6q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^k_{3,4} &= 552\\
p^k_{3,4}(q) &= 4q^{24} + 9q^{23} + 19q^{22} + 26q^{21} + 35q^{20} + 39q^{19}\\
&\qquad + 46q^{18} + 46q^{17} + 48q^{16} + 45q^{15} + 43q^{14} + 37q^{13}\\
&\qquad + 34q^{12} + 27q^{11} + 23q^{10} + 18q^9 + 15q^8 + 11q^7\\
&\qquad + 9q^6 + 6q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^k_{4,4} &= 1425\\
p^k_{4,4}(q) &= 5q^{32} + 12q^{31} + 26q^{30} + 38q^{29} + 54q^{28} + 64q^{27} + 77q^{26} + 84q^{25}\\
&\qquad + 93q^{24} + 94q^{23} + 98q^{22} + 94q^{21} + 92q^{20} + 85q^{19}\\
&\qquad + 80q^{18} + 70q^{17} + 64q^{16} + 54q^{15} + 47q^{14} + 39q^{13}\\
&\qquad + 34q^{12} + 27q^{11} + 23q^{10} + 18q^9 + 15q^8 + 11q^7\\
&\qquad + 9q^6 + 6q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi^k_{5,4} &= 3066\\
p^k_{5,4}(q) &= 6q^{40} + 15q^{39} + 33q^{38} + 50q^{37}\\
&\qquad + 73q^{36} + 91q^{35} + 112q^{34} + 126q^{33} + 144q^{32} + 153q^{31}\\
&\qquad + 165q^{30} + 168q^{29} + 173q^{28} + 169q^{27} + 168q^{26} + 159q^{25}\\
&\qquad + 153q^{24} + 141q^{23} + 132q^{22} + 118q^{21} + 108q^{20} + 94q^{19}\\
&\qquad + 84q^{18} + 72q^{17} + 64q^{16} + 54q^{15} + 47q^{14} + 39q^{13}\\
&\qquad + 34q^{12} + 27q^{11} + 23q^{10} + 18q^9 + 15q^8 + 11q^7\\
&\qquad + 9q^6 + 6q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}
$$

EXAMPLE C.38. For $N \in [5]$ and $n = 4$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,4} &= 19\\
p_{1,4}(q) &= 1q^8 + 1q^7 + 3q^6 + 3q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{2,4} &= 85\\
p_{2,4}(q) &= 1q^{16} + 1q^{15} + 3q^{14} + 4q^{13} + 7q^{12} + 8q^{11} + 10q^{10} + 10q^9 + 10q^8 + 8q^7\\
&\quad + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{3,4} &= 231\\
p_{3,4}(q) &= 1q^{24} + 1q^{23} + 3q^{22} + 4q^{21} + 7q^{20} + 9q^{19}\\
&\quad + 13q^{18} + 15q^{17} + 18q^{16} + 19q^{15} + 20q^{14} + 19q^{13}\\
&\quad + 19q^{12} + 16q^{11} + 14q^{10} + 12q^9 + 10q^8 + 8q^7\\
&\quad + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{4,4} &= 489\\
p_{4,4}(q) &= 1q^{32} + 1q^{31} + 3q^{30} + 4q^{29} + 7q^{28} + 9q^{27} + 13q^{26} + 16q^{25}\\
&\quad + 21q^{24} + 24q^{23} + 28q^{22} + 30q^{21} + 32q^{20} + 32q^{19}\\
&\quad + 33q^{18} + 31q^{17} + 30q^{16} + 27q^{15} + 24q^{14} + 21q^{13}\\
&\quad + 19q^{12} + 16q^{11} + 14q^{10} + 12q^9 + 10q^8 + 8q^7\\
&\quad + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{5,4} &= 891\\
p_{5,4}(q) &= 1q^{40} + 1q^{39} + 3q^{38} + 4q^{37}\\
&\quad + 7q^{36} + 9q^{35} + 13q^{34} + 16q^{33} + 21q^{32} + 25q^{31}\\
&\quad + 31q^{30} + 35q^{29} + 40q^{28} + 43q^{27} + 46q^{26} + 47q^{25}\\
&\quad + 49q^{24} + 48q^{23} + 48q^{22} + 46q^{21} + 44q^{20} + 40q^{19}\\
&\quad + 37q^{18} + 33q^{17} + 30q^{16} + 27q^{15} + 24q^{14} + 21q^{13}\\
&\quad + 19q^{12} + 16q^{11} + 14q^{10} + 12q^9 + 10q^8 + 8q^7\\
&\quad + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}$$

C.2.5. Poincaré Polynomials for n=5.

EXAMPLE C.39. For $N \in [5]$ and $n = 5$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^a(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,5}^a &= 252 \\
p_{1,5}^a(q) &= 1q^{25} + 1q^{24} + 2q^{23} + 3q^{22} + 5q^{21} + 7q^{20} + 9q^{19} \\
&\quad + 11q^{18} + 14q^{17} + 16q^{16} + 18q^{15} + 19q^{14} + 20q^{13} \\
&\quad + 20q^{12} + 19q^{11} + 18q^{10} + 16q^{9} + 14q^{8} + 11q^{7} \\
&\quad + 9q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0} \\
\chi_{2,5}^a &= 8953 \\
p_{2,5}^a(q) &= 1q^{50} + 1q^{49} + 3q^{48} + 5q^{47} + 10q^{46} + 16q^{45} + 26q^{44} + 37q^{43} \\
&\quad + 56q^{42} + 75q^{41} + 103q^{40} + 131q^{39} + 168q^{38} + 203q^{37} \\
&\quad + 247q^{36} + 285q^{35} + 329q^{34} + 365q^{33} + 403q^{32} + 429q^{31} \\
&\quad + 457q^{30} + 468q^{29} + 479q^{28} + 475q^{27} + 469q^{26} + 450q^{25} \\
&\quad + 431q^{24} + 399q^{23} + 370q^{22} + 333q^{21} + 299q^{20} + 260q^{19} \\
&\quad + 228q^{18} + 192q^{17} + 163q^{16} + 133q^{15} + 109q^{14} + 86q^{13} \\
&\quad + 69q^{12} + 52q^{11} + 41q^{10} + 30q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0} \\
\chi_{3,5}^a &= 116304 \\
p_{3,5}^a(q) &= 1q^{75} + 1q^{74} + 3q^{73} + 6q^{72} + 12q^{71} + 21q^{70} + 37q^{69} + 58q^{68} + 92q^{67} \\
&\quad + 137q^{66} + 200q^{65} + 279q^{64} + 385q^{63} + 510q^{62} + 667q^{61} \\
&\quad + 849q^{60} + 1061q^{59} + 1297q^{58} + 1564q^{57} + 1845q^{56} + 2149q^{55} \\
&\quad + 2460q^{54} + 2777q^{53} + 3088q^{52} + 3394q^{51} + 3675q^{50} + 3935q^{49} \\
&\quad + 4162q^{48} + 4351q^{47} + 4498q^{46} + 4604q^{45} + 4659q^{44} + 4670q^{43} \\
&\quad + 4637q^{42} + 4557q^{41} + 4440q^{40} + 4287q^{39} + 4102q^{38} + 3891q^{37} \\
&\quad + 3663q^{36} + 3415q^{35} + 3161q^{34} + 2901q^{33} + 2642q^{32} + 2385q^{31} \\
&\quad + 2141q^{30} + 1901q^{29} + 1679q^{28} + 1469q^{27} + 1277q^{26} + 1099q^{25} \\
&\quad + 942q^{24} + 797q^{23} + 672q^{22} + 560q^{21} + 464q^{20} + 379q^{19} \\
&\quad + 310q^{18} + 248q^{17} + 199q^{16} + 157q^{15} + 123q^{14} + 94q^{13} \\
&\quad + 73q^{12} + 54q^{11} + 41q^{10} + 30q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}$$

$$
\begin{aligned}
\chi^k_{4,5} &= 856945\\
p^k_{4,5}(q) &= 1q^{100} + 1q^{99} + 3q^{98} + 6q^{97} + 13q^{96} + 23q^{95} + 42q^{94} + 69q^{93} + 115q^{92} + 177q^{91}\\
&\quad + 272q^{90} + 396q^{89} + 573q^{88} + 795q^{87} + 1092q^{86} + 1454q^{85}\\
&\quad + 1914q^{84} + 2454q^{83} + 3113q^{82} + 3864q^{81} + 4746q^{80} + 5719q^{79}\\
&\quad + 6825q^{78} + 8012q^{77} + 9319q^{76} + 10682q^{75} + 12137q^{74} + 13613q^{73}\\
&\quad + 15146q^{72} + 16651q^{71} + 18169q^{70} + 19613q^{69} + 21021q^{68} + 22307q^{67}\\
&\quad + 23517q^{66} + 24564q^{65} + 25500q^{64} + 26245q^{63} + 26854q^{62} + 27257q^{61}\\
&\quad + 27518q^{60} + 27566q^{59} + 27475q^{58} + 27187q^{57} + 26771q^{56} + 26178q^{55}\\
&\quad + 25483q^{54} + 24638q^{53} + 23720q^{52} + 22687q^{51} + 21609q^{50} + 20451q^{49}\\
&\quad + 19282q^{48} + 18062q^{47} + 16858q^{46} + 15637q^{45} + 14452q^{44} + 13273q^{43}\\
&\quad + 12151q^{42} + 11052q^{41} + 10022q^{40} + 9029q^{39} + 8109q^{38} + 7235q^{37}\\
&\quad + 6439q^{36} + 5689q^{35} + 5014q^{34} + 4388q^{33} + 3830q^{32} + 3317q^{31}\\
&\quad + 2868q^{30} + 2458q^{29} + 2104q^{28} + 1785q^{27} + 1511q^{26} + 1267q^{25}\\
&\quad + 1062q^{24} + 879q^{23} + 728q^{22} + 596q^{21} + 488q^{20} + 393q^{19}\\
&\quad + 318q^{18} + 252q^{17} + 201q^{16} + 157q^{15} + 123q^{14} + 94q^{13}\\
&\quad + 73q^{12} + 54q^{11} + 41q^{10} + 30q^{9} + 22q^{8} + 15q^{7}\\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

$$
\begin{aligned}
\chi^k_{5,5} &= 4395456 \\
p^k_{5,5}(q) &= 1q^{125} + 1q^{124} + 3q^{123} + 6q^{122} + 13q^{121} \\
&\quad + 24q^{120} + 44q^{119} + 74q^{118} + 126q^{117} + 200q^{116} + 314q^{115} \\
&\quad + 472q^{114} + 700q^{113} + 1003q^{112} + 1417q^{111} + 1949q^{110} + 2641q^{109} \\
&\quad + 3503q^{108} + 4580q^{107} + 5880q^{106} + 7456q^{105} + 9301q^{104} + 11467q^{103} \\
&\quad + 13946q^{102} + 16776q^{101} + 19936q^{100} + 23458q^{99} + 27301q^{98} + 31487q^{97} \\
&\quad + 35962q^{96} + 40725q^{95} + 45713q^{94} + 50919q^{93} + 56254q^{92} + 61703q^{91} \\
&\quad + 67181q^{90} + 72652q^{89} + 78032q^{88} + 83289q^{87} + 88331q^{86} + 93133q^{85} \\
&\quad + 97616q^{84} + 101746q^{83} + 105468q^{82} + 108762q^{81} + 111571q^{80} + 113891q^{79} \\
&\quad + 115695q^{78} + 116974q^{77} + 117724q^{76} + 117953q^{75} + 117661q^{74} + 116872q^{73} \\
&\quad + 115607q^{72} + 113881q^{71} + 111739q^{70} + 109206q^{69} + 106319q^{68} + 103113q^{67} \\
&\quad + 99640q^{66} + 95921q^{65} + 92017q^{64} + 87951q^{63} + 83775q^{62} + 79514q^{61} \\
&\quad + 75222q^{60} + 70905q^{59} + 66625q^{58} + 62387q^{57} + 58232q^{56} + 54166q^{55} \\
&\quad + 50232q^{54} + 46419q^{53} + 42769q^{52} + 39269q^{51} + 35948q^{50} + 32793q^{49} \\
&\quad + 29830q^{48} + 27033q^{47} + 24432q^{46} + 22001q^{45} + 19754q^{44} + 17670q^{43} \\
&\quad + 15764q^{42} + 14006q^{41} + 12412q^{40} + 10954q^{39} + 9640q^{38} + 8448q^{37} \\
&\quad + 7385q^{36} + 6423q^{35} + 5574q^{34} + 4814q^{33} + 4146q^{32} + 3551q^{31} \\
&\quad + 3036q^{30} + 2578q^{29} + 2186q^{28} + 1841q^{27} + 1547q^{26} + 1291q^{25} \\
&\quad + 1076q^{24} + 887q^{23} + 732q^{22} + 598q^{21} + 488q^{20} + 393q^{19} \\
&\quad + 318q^{18} + 252q^{17} + 201q^{16} + 157q^{15} + 123q^{14} + 94q^{13} \\
&\quad + 73q^{12} + 54q^{11} + 41q^{10} + 30q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

EXAMPLE C.40. For $N \in [5]$, $n = 5$ and $k = 4$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,5}^k &= 182\\
p_{1,5}^k(q) &= 2q^{20} + 2q^{19} + 4q^{18} + 6q^{17} + 9q^{16} + 11q^{15} + 14q^{14} + 15q^{13}\\
&\quad + 17q^{12} + 17q^{11} + 17q^{10} + 15q^{9} + 14q^{8} + 11q^{7}\\
&\quad + 9q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{2,5}^k &= 4707\\
p_{2,5}^k(q) &= 3q^{40} + 4q^{39} + 10q^{38} + 17q^{37}\\
&\quad + 31q^{36} + 45q^{35} + 69q^{34} + 91q^{33} + 124q^{32} + 152q^{31}\\
&\quad + 188q^{30} + 214q^{29} + 247q^{28} + 265q^{27} + 286q^{26} + 292q^{25}\\
&\quad + 298q^{24} + 289q^{23} + 283q^{22} + 263q^{21} + 246q^{20} + 219q^{19}\\
&\quad + 197q^{18} + 169q^{17} + 148q^{16} + 122q^{15} + 102q^{14} + 81q^{13}\\
&\quad + 66q^{12} + 50q^{11} + 40q^{10} + 29q^{9} + 22q^{8} + 15q^{7}\\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{3,5}^k &= 47660\\
p_{3,5}^k(q) &= 4q^{60} + 6q^{59} + 16q^{58} + 30q^{57} + 57q^{56} + 91q^{55}\\
&\quad + 150q^{54} + 217q^{53} + 316q^{52} + 428q^{51} + 572q^{50} + 723q^{49}\\
&\quad + 906q^{48} + 1082q^{47} + 1280q^{46} + 1463q^{45} + 1650q^{44} + 1806q^{43}\\
&\quad + 1960q^{42} + 2069q^{41} + 2167q^{40} + 2220q^{39} + 2253q^{38} + 2241q^{37}\\
&\quad + 2217q^{36} + 2149q^{35} + 2072q^{34} + 1964q^{33} + 1850q^{32} + 1714q^{31}\\
&\quad + 1584q^{30} + 1438q^{29} + 1302q^{28} + 1160q^{27} + 1030q^{26} + 900q^{25}\\
&\quad + 787q^{24} + 674q^{23} + 578q^{22} + 487q^{21} + 410q^{20} + 338q^{19}\\
&\quad + 280q^{18} + 226q^{17} + 184q^{16} + 146q^{15} + 116q^{14} + 89q^{13}\\
&\quad + 70q^{12} + 52q^{11} + 40q^{10} + 29q^{9} + 22q^{8} + 15q^{7}\\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}$$

$$
\begin{aligned}
\chi^k_{4,5} &= 287005 \\
p^k_{4,5}(q) &= 5q^{80} + 8q^{79} + 22q^{78} + 43q^{77} + 85q^{76} + 141q^{75} + 241q^{74} + 366q^{73} \\
&\quad + 557q^{72} + 789q^{71} + 1105q^{70} + 1469q^{69} + 1935q^{68} + 2441q^{67} \\
&\quad + 3045q^{66} + 3679q^{65} + 4392q^{64} + 5102q^{63} + 5870q^{62} + 6596q^{61} \\
&\quad + 7345q^{60} + 8019q^{59} + 8681q^{58} + 9238q^{57} + 9761q^{56} + 10152q^{55} \\
&\quad + 10491q^{54} + 10695q^{53} + 10838q^{52} + 10844q^{51} + 10797q^{50} + 10622q^{49} \\
&\quad + 10403q^{48} + 10078q^{47} + 9722q^{46} + 9282q^{45} + 8833q^{44} + 8318q^{43} \\
&\quad + 7811q^{42} + 7264q^{41} + 6738q^{40} + 6189q^{39} + 5675q^{38} + 5152q^{37} \\
&\quad + 4670q^{36} + 4191q^{35} + 3756q^{34} + 3333q^{33} + 2957q^{32} + 2593q^{31} \\
&\quad + 2274q^{30} + 1972q^{29} + 1711q^{28} + 1467q^{27} + 1259q^{26} + 1066q^{25} \\
&\quad + 905q^{24} + 756q^{23} + 634q^{22} + 523q^{21} + 434q^{20} + 352q^{19} \\
&\quad + 288q^{18} + 230q^{17} + 186q^{16} + 146q^{15} + 116q^{14} + 89q^{13} \\
&\quad + 70q^{12} + 52q^{11} + 40q^{10} + 29q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0} \\
\chi^k_{5,5} &= 1243476 \\
p^k_{5,5}(q) &= 6q^{100} + 10q^{99} + 28q^{98} + 56q^{97} \\
&\quad + 113q^{96} + 193q^{95} + 336q^{94} + 525q^{93} + 818q^{92} + 1194q^{91} \\
&\quad + 1718q^{90} + 2357q^{89} + 3195q^{88} + 4169q^{87} + 5369q^{86} + 6721q^{85} \\
&\quad + 8300q^{84} + 10009q^{83} + 11933q^{82} + 13941q^{81} + 16116q^{80} + 18315q^{79} \\
&\quad + 20612q^{78} + 22858q^{77} + 25136q^{76} + 27281q^{75} + 29382q^{74} + 31289q^{73} \\
&\quad + 33087q^{72} + 34631q^{71} + 36027q^{70} + 37128q^{69} + 38049q^{68} + 38666q^{67} \\
&\quad + 39088q^{66} + 39208q^{65} + 39147q^{64} + 38799q^{63} + 38289q^{62} + 37533q^{61} \\
&\quad + 36644q^{60} + 35544q^{59} + 34356q^{58} + 33002q^{57} + 31596q^{56} + 30074q^{55} \\
&\quad + 28534q^{54} + 26920q^{53} + 25329q^{52} + 23697q^{51} + 22113q^{50} + 20523q^{49} \\
&\quad + 19000q^{48} + 17493q^{47} + 16073q^{46} + 14686q^{45} + 13390q^{44} + 12141q^{43} \\
&\quad + 10987q^{42} + 9886q^{41} + 8883q^{40} + 7932q^{39} + 7072q^{38} + 6268q^{37} \\
&\quad + 5548q^{36} + 4878q^{35} + 4285q^{34} + 3738q^{33} + 3259q^{32} + 2819q^{31} \\
&\quad + 2438q^{30} + 2090q^{29} + 1793q^{28} + 1523q^{27} + 1295q^{26} + 1090q^{25} \\
&\quad + 919q^{24} + 764q^{23} + 638q^{22} + 525q^{21} + 434q^{20} + 352q^{19} \\
&\quad + 288q^{18} + 230q^{17} + 186q^{16} + 146q^{15} + 116q^{14} + 89q^{13} \\
&\quad + 70q^{12} + 52q^{11} + 40q^{10} + 29q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

EXAMPLE C.41. For $N \in [5]$, $n = 5$ and $k = 3$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n)$ are given by

$$\chi_{1,5}^k = 132$$

$$\begin{aligned} p_{1,5}^k(q) = {} & 1q^{17} + 2q^{16} + 5q^{15} + 7q^{14} + 10q^{13} \\ & + 12q^{12} + 14q^{11} + 15q^{10} + 14q^9 + 13q^8 + 11q^7 \\ & + 9q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \end{aligned}$$

$$\chi_{2,5}^k = 2495$$

$$\begin{aligned} p_{2,5}^k(q) = {} & 1q^{34} + 2q^{33} + 7q^{32} + 15q^{31} \\ & + 30q^{30} + 45q^{29} + 70q^{28} + 92q^{27} + 118q^{26} + 140q^{25} \\ & + 161q^{24} + 171q^{23} + 182q^{22} + 181q^{21} + 179q^{20} + 168q^{19} \\ & + 157q^{18} + 139q^{17} + 125q^{16} + 107q^{15} + 91q^{14} + 74q^{13} \\ & + 61q^{12} + 47q^{11} + 38q^{10} + 28q^9 + 21q^8 + 15q^7 \\ & + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \end{aligned}$$

$$\chi_{3,5}^k = 19748$$

$$\begin{aligned} p_{3,5}^k(q) = {} & 1q^{51} + 2q^{50} + 7q^{49} + 17q^{48} + 38q^{47} + 70q^{46} + 124q^{45} + 189q^{44} + 277q^{43} \\ & + 376q^{42} + 486q^{41} + 598q^{40} + 712q^{39} + 812q^{38} + 904q^{37} \\ & + 979q^{36} + 1032q^{35} + 1064q^{34} + 1078q^{33} + 1070q^{32} + 1044q^{31} \\ & + 1005q^{30} + 950q^{29} + 889q^{28} + 821q^{27} + 749q^{26} + 674q^{25} \\ & + 602q^{24} + 529q^{23} + 462q^{22} + 398q^{21} + 340q^{20} + 286q^{19} \\ & + 240q^{18} + 197q^{17} + 162q^{16} + 131q^{15} + 105q^{14} + 82q^{13} \\ & + 65q^{12} + 49q^{11} + 38q^{10} + 28q^9 + 21q^8 + 15q^7 \\ & + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \end{aligned}$$

$$\chi_{4,5}^k = 97401$$

$$\begin{aligned} p_{4,5}^k(q) = {} & 1q^{68} + 2q^{67} + 7q^{66} + 17q^{65} + 40q^{64} + 78q^{63} + 149q^{62} + 250q^{61} \\ & + 403q^{60} + 594q^{59} + 842q^{58} + 1123q^{57} + 1451q^{56} + 1792q^{55} \\ & + 2157q^{54} + 2511q^{53} + 2868q^{52} + 3189q^{51} + 3490q^{50} + 3735q^{49} \\ & + 3947q^{48} + 4092q^{47} + 4197q^{46} + 4235q^{45} + 4238q^{44} + 4179q^{43} \\ & + 4090q^{42} + 3953q^{41} + 3798q^{40} + 3606q^{39} + 3407q^{38} + 3184q^{37} \\ & + 2963q^{36} + 2728q^{35} + 2502q^{34} + 2271q^{33} + 2055q^{32} + 1840q^{31} \\ & + 1643q^{30} + 1451q^{29} + 1279q^{28} + 1115q^{27} + 971q^{26} + 836q^{25} \\ & + 719q^{24} + 610q^{23} + 518q^{22} + 434q^{21} + 364q^{20} + 300q^{19} \\ & + 248q^{18} + 201q^{17} + 164q^{16} + 131q^{15} + 105q^{14} + 82q^{13} \\ & + 65q^{12} + 49q^{11} + 38q^{10} + 28q^9 + 21q^8 + 15q^7 \\ & + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \end{aligned}$$

$$\begin{aligned}
\chi^k_{5,5} &= 357036 \\
p^k_{5,5}(q) &= 1q^{85} + 2q^{84} + 7q^{83} + 17q^{82} + 40q^{81} + 80q^{80} + 157q^{79} \\
&\quad + 275q^{78} + 464q^{77} + 729q^{76} + 1097q^{75} + 1556q^{74} + 2134q^{73} \\
&\quad + 2799q^{72} + 3561q^{71} + 4390q^{70} + 5278q^{69} + 6187q^{68} + 7122q^{67} \\
&\quad + 8035q^{66} + 8920q^{65} + 9749q^{64} + 10512q^{63} + 11180q^{62} + 11756q^{61} \\
&\quad + 12218q^{60} + 12567q^{59} + 12805q^{58} + 12927q^{57} + 12940q^{56} + 12855q^{55} \\
&\quad + 12674q^{54} + 12404q^{53} + 12065q^{52} + 11658q^{51} + 11201q^{50} + 10699q^{49} \\
&\quad + 10167q^{48} + 9605q^{47} + 9034q^{46} + 8450q^{45} + 7868q^{44} + 7289q^{43} \\
&\quad + 6724q^{42} + 6169q^{41} + 5640q^{40} + 5128q^{39} + 4645q^{38} + 4187q^{37} \\
&\quad + 3761q^{36} + 3359q^{35} + 2992q^{34} + 2650q^{33} + 2340q^{32} + 2055q^{31} \\
&\quad + 1800q^{30} + 1565q^{29} + 1359q^{28} + 1171q^{27} + 1007q^{26} + 860q^{25} \\
&\quad + 733q^{24} + 618q^{23} + 522q^{22} + 436q^{21} + 364q^{20} + 300q^{19} \\
&\quad + 248q^{18} + 201q^{17} + 164q^{16} + 131q^{15} + 105q^{14} + 82q^{13} \\
&\quad + 65q^{12} + 49q^{11} + 38q^{10} + 28q^{9} + 21q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.42. For $N \in [5]$, $n = 5$ and $k = 2$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,5}^k &= 96\\
p_{1,5}^k(q) &= 2q^{14} + 4q^{13} + 7q^{12} + 9q^{11} + 12q^{10} + 12q^9 + 12q^8 + 10q^7\\
&\quad + 9q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{2,5}^k &= 1329\\
p_{2,5}^k(q) &= 3q^{28} + 8q^{27} + 20q^{26} + 32q^{25}\\
&\quad + 52q^{24} + 67q^{23} + 86q^{22} + 96q^{21} + 107q^{20} + 107q^{19}\\
&\quad + 109q^{18} + 101q^{17} + 96q^{16} + 85q^{15} + 76q^{14} + 63q^{13}\\
&\quad + 54q^{12} + 42q^{11} + 35q^{10} + 26q^9 + 20q^8 + 14q^7\\
&\quad + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{3,5}^k &= 8234\\
p_{3,5}^k(q) &= 4q^{42} + 12q^{41} + 33q^{40} + 63q^{39} + 108q^{38} + 158q^{37}\\
&\quad + 222q^{36} + 280q^{35} + 344q^{34} + 394q^{33} + 441q^{32} + 469q^{31}\\
&\quad + 493q^{30} + 495q^{29} + 495q^{28} + 479q^{27} + 461q^{26} + 431q^{25}\\
&\quad + 403q^{24} + 365q^{23} + 331q^{22} + 292q^{21} + 258q^{20} + 221q^{19}\\
&\quad + 191q^{18} + 159q^{17} + 134q^{16} + 110q^{15} + 90q^{14} + 71q^{13}\\
&\quad + 58q^{12} + 44q^{11} + 35q^{10} + 26q^9 + 20q^8 + 14q^7\\
&\quad + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0\\
\chi_{4,5}^k &= 33293\\
p_{4,5}^k(q) &= 5q^{56} + 16q^{55}\\
&\quad + 46q^{54} + 94q^{53} + 174q^{52} + 269q^{51} + 398q^{50} + 534q^{49}\\
&\quad + 697q^{48} + 850q^{47} + 1016q^{46} + 1156q^{45} + 1296q^{44} + 1398q^{43}\\
&\quad + 1493q^{42} + 1545q^{41} + 1589q^{40} + 1592q^{39} + 1589q^{38} + 1552q^{37}\\
&\quad + 1514q^{36} + 1447q^{35} + 1382q^{34} + 1296q^{33} + 1215q^{32} + 1118q^{31}\\
&\quad + 1031q^{30} + 932q^{29} + 845q^{28} + 752q^{27} + 671q^{26} + 588q^{25}\\
&\quad + 518q^{24} + 446q^{23} + 387q^{22} + 329q^{21} + 282q^{20} + 235q^{19}\\
&\quad + 199q^{18} + 163q^{17} + 136q^{16} + 110q^{15} + 90q^{14} + 71q^{13}\\
&\quad + 58q^{12} + 44q^{11} + 35q^{10} + 26q^9 + 20q^8 + 14q^7\\
&\quad + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}$$

$$
\begin{aligned}
\chi^k_{5,5} &= 103320 \\
p^k_{5,5}(q) &= 6q^{70} + 20q^{69} + 59q^{68} + 125q^{67} \\
&\quad + 240q^{66} + 392q^{65} + 598q^{64} + 836q^{63} + 1126q^{62} + 1432q^{61} \\
&\quad + 1777q^{60} + 2113q^{59} + 2466q^{58} + 2786q^{57} + 3100q^{56} + 3362q^{55} \\
&\quad + 3607q^{54} + 3787q^{53} + 3942q^{52} + 4032q^{51} + 4097q^{50} + 4101q^{49} \\
&\quad + 4087q^{48} + 4019q^{47} + 3940q^{46} + 3818q^{45} + 3688q^{44} + 3524q^{43} \\
&\quad + 3361q^{42} + 3170q^{41} + 2986q^{40} + 2783q^{39} + 2590q^{38} + 2386q^{37} \\
&\quad + 2197q^{36} + 2001q^{35} + 1823q^{34} + 1644q^{33} + 1482q^{32} + 1322q^{31} \\
&\quad + 1182q^{30} + 1043q^{29} + 923q^{28} + 807q^{27} + 707q^{26} + 612q^{25} \\
&\quad + 532q^{24} + 454q^{23} + 391q^{22} + 331q^{21} + 282q^{20} + 235q^{19} \\
&\quad + 199q^{18} + 163q^{17} + 136q^{16} + 110q^{15} + 90q^{14} + 71q^{13} \\
&\quad + 58q^{12} + 44q^{11} + 35q^{10} + 26q^{9} + 20q^{8} + 14q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

EXAMPLE C.43. For $N \in [5]$, $n = 5$ and $k = 1$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,5}^k &= 70 \\
p_{1,5}^k(q) &= 1q^{13} + 2q^{12} + 5q^{11} + 7q^{10} + 9q^9 + 10q^8 + 9q^7 \\
&\quad + 8q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi_{2,5}^k &= 711 \\
p_{2,5}^k(q) &= 1q^{26} + 2q^{25} + 7q^{24} + 13q^{23} + 24q^{22} + 33q^{21} + 45q^{20} + 52q^{19} \\
&\quad + 59q^{18} + 60q^{17} + 62q^{16} + 58q^{15} + 55q^{14} + 49q^{13} \\
&\quad + 43q^{12} + 35q^{11} + 30q^{10} + 23q^9 + 18q^8 + 13q^7 \\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi_{3,5}^k &= 3452 \\
p_{3,5}^k(q) &= 1q^{39} + 2q^{38} + 7q^{37} + 15q^{36} + 30q^{35} + 49q^{34} + 76q^{33} + 102q^{32} + 131q^{31} \\
&\quad + 157q^{30} + 179q^{29} + 196q^{28} + 209q^{27} + 215q^{26} + 216q^{25} \\
&\quad + 214q^{24} + 206q^{23} + 195q^{22} + 182q^{21} + 166q^{20} + 149q^{19} \\
&\quad + 133q^{18} + 115q^{17} + 99q^{16} + 84q^{15} + 70q^{14} + 57q^{13} \\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^9 + 18q^8 + 13q^7 \\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0 \\
\chi_{4,5}^k &= 11451 \\
p_{4,5}^k(q) &= 1q^{52} + 2q^{51} + 7q^{50} + 15q^{49} \\
&\quad + 32q^{48} + 55q^{47} + 92q^{46} + 134q^{45} + 189q^{44} + 243q^{43} \\
&\quad + 304q^{42} + 358q^{41} + 413q^{40} + 456q^{39} + 498q^{38} + 525q^{37} \\
&\quad + 549q^{36} + 560q^{35} + 566q^{34} + 560q^{33} + 553q^{32} + 533q^{31} \\
&\quad + 512q^{30} + 484q^{29} + 455q^{28} + 420q^{27} + 388q^{26} + 351q^{25} \\
&\quad + 318q^{24} + 283q^{23} + 251q^{22} + 219q^{21} + 192q^{20} + 164q^{19} \\
&\quad + 141q^{18} + 119q^{17} + 101q^{16} + 84q^{15} + 70q^{14} + 57q^{13} \\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^9 + 18q^8 + 13q^7 \\
&\quad + 10q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + 1q^1 + 1q^0
\end{aligned}$$

$$
\begin{aligned}
\chi^k_{5,5} &= 30102 \\
p^k_{5,5}(q) &= 1q^{65} + 2q^{64} + 7q^{63} + 15q^{62} + 32q^{61} \\
&\quad + 57q^{60} + 98q^{59} + 150q^{58} + 221q^{57} + 302q^{56} + 398q^{55} \\
&\quad + 497q^{54} + 603q^{53} + 705q^{52} + 805q^{51} + 896q^{50} + 980q^{49} \\
&\quad + 1051q^{48} + 1111q^{47} + 1159q^{46} + 1193q^{45} + 1214q^{44} + 1224q^{43} \\
&\quad + 1221q^{42} + 1207q^{41} + 1185q^{40} + 1151q^{39} + 1110q^{38} + 1064q^{37} \\
&\quad + 1011q^{36} + 953q^{35} + 895q^{34} + 833q^{33} + 771q^{32} + 709q^{31} \\
&\quad + 648q^{30} + 588q^{29} + 532q^{28} + 476q^{27} + 425q^{26} + 377q^{25} \\
&\quad + 333q^{24} + 291q^{23} + 255q^{22} + 221q^{21} + 192q^{20} + 164q^{19} \\
&\quad + 141q^{18} + 119q^{17} + 101q^{16} + 84q^{15} + 70q^{14} + 57q^{13} \\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^{9} + 18q^{8} + 13q^{7} \\
&\quad + 10q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

EXAMPLE C.44. For $N \in [5]$ and $n = 5$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N(\widehat{\mathfrak{gl}}_n)$ are given by

$$
\begin{aligned}
\chi_{1,5} &= 51\\
p_{1,5}(q) &= 1q^{12} + 2q^{11} + 4q^{10} + 5q^{9} + 7q^{8} + 7q^{7}\\
&\quad + 7q^{6} + 6q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{2,5} &= 381\\
p_{2,5}(q) &= 1q^{24} + 2q^{23} + 6q^{22} + 9q^{21} + 15q^{20} + 19q^{19}\\
&\quad + 25q^{18} + 27q^{17} + 32q^{16} + 32q^{15} + 33q^{14} + 31q^{13}\\
&\quad + 30q^{12} + 25q^{11} + 23q^{10} + 18q^{9} + 15q^{8} + 11q^{7}\\
&\quad + 9q^{6} + 6q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{3,5} &= 1451\\
p_{3,5}(q) &= 1q^{36} + 2q^{35} + 6q^{34} + 11q^{33} + 19q^{32} + 27q^{31}\\
&\quad + 39q^{30} + 47q^{29} + 58q^{28} + 66q^{27} + 75q^{26} + 80q^{25}\\
&\quad + 87q^{24} + 88q^{23} + 90q^{22} + 88q^{21} + 86q^{20} + 80q^{19}\\
&\quad + 76q^{18} + 68q^{17} + 62q^{16} + 54q^{15} + 47q^{14} + 39q^{13}\\
&\quad + 34q^{12} + 27q^{11} + 23q^{10} + 18q^{9} + 15q^{8} + 11q^{7}\\
&\quad + 9q^{6} + 6q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{4,5} &= 3951\\
p_{4,5}(q) &= 1q^{48} + 2q^{47} + 6q^{46} + 11q^{45} + 21q^{44} + 31q^{43}\\
&\quad + 47q^{42} + 61q^{41} + 80q^{40} + 94q^{39} + 113q^{38} + 126q^{37}\\
&\quad + 143q^{36} + 154q^{35} + 168q^{34} + 175q^{33} + 185q^{32} + 187q^{31}\\
&\quad + 191q^{30} + 188q^{29} + 187q^{28} + 179q^{27} + 174q^{26} + 163q^{25}\\
&\quad + 155q^{24} + 142q^{23} + 132q^{22} + 118q^{21} + 108q^{20} + 94q^{19}\\
&\quad + 84q^{18} + 72q^{17} + 64q^{16} + 54q^{15} + 47q^{14} + 39q^{13}\\
&\quad + 34q^{12} + 27q^{11} + 23q^{10} + 18q^{9} + 15q^{8} + 11q^{7}\\
&\quad + 9q^{6} + 6q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{5,5} &= 8801\\
p_{5,5}(q) &= 1q^{60} + 2q^{59} + 6q^{58} + 11q^{57} + 21q^{56} + 33q^{55}\\
&\quad + 51q^{54} + 69q^{53} + 94q^{52} + 116q^{51} + 143q^{50} + 166q^{49}\\
&\quad + 193q^{48} + 214q^{47} + 240q^{46} + 259q^{45} + 281q^{44} + 297q^{43}\\
&\quad + 315q^{42} + 325q^{41} + 338q^{40} + 342q^{39} + 347q^{38} + 345q^{37}\\
&\quad + 344q^{36} + 335q^{35} + 329q^{34} + 316q^{33} + 305q^{32} + 289q^{31}\\
&\quad + 275q^{30} + 256q^{29} + 241q^{28} + 221q^{27} + 204q^{26} + 185q^{25}\\
&\quad + 169q^{24} + 150q^{23} + 136q^{22} + 120q^{21} + 108q^{20} + 94q^{19}\\
&\quad + 84q^{18} + 72q^{17} + 64q^{16} + 54q^{15} + 47q^{14} + 39q^{13}\\
&\quad + 34q^{12} + 27q^{11} + 23q^{10} + 18q^{9} + 15q^{8} + 11q^{7}\\
&\quad + 9q^{6} + 6q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

C.2.6. Poincaré Polynomials for n=6.

EXAMPLE C.45. For $N \in [3]$ and $n = 6$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^a(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,6}^a &= 924 \\
p_{1,6}^a(q) &= 1q^{36} + 1q^{35} + 2q^{34} + 3q^{33} + 5q^{32} + 7q^{31} \\
&\quad + 11q^{30} + 13q^{29} + 18q^{28} + 22q^{27} + 28q^{26} + 32q^{25} \\
&\quad + 39q^{24} + 42q^{23} + 48q^{22} + 51q^{21} + 55q^{20} + 55q^{19} \\
&\quad + 58q^{18} + 55q^{17} + 55q^{16} + 51q^{15} + 48q^{14} + 42q^{13} \\
&\quad + 39q^{12} + 32q^{11} + 28q^{10} + 22q^{9} + 18q^{8} + 13q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0} \\
\chi_{2,6}^a &= 73789 \\
p_{2,6}^a(q) &= 1q^{72} + 1q^{71} + 3q^{70} + 5q^{69} + 10q^{68} + 16q^{67} \\
&\quad + 28q^{66} + 41q^{65} + 64q^{64} + 91q^{63} + 131q^{62} + 178q^{61} \\
&\quad + 244q^{60} + 316q^{59} + 412q^{58} + 518q^{57} + 648q^{56} + 786q^{55} \\
&\quad + 951q^{54} + 1118q^{53} + 1310q^{52} + 1499q^{51} + 1704q^{50} + 1898q^{49} \\
&\quad + 2104q^{48} + 2284q^{47} + 2467q^{46} + 2619q^{45} + 2762q^{44} + 2865q^{43} \\
&\quad + 2957q^{42} + 3001q^{41} + 3031q^{40} + 3015q^{39} + 2982q^{38} + 2907q^{37} \\
&\quad + 2822q^{36} + 2696q^{35} + 2565q^{34} + 2407q^{33} + 2248q^{32} + 2069q^{31} \\
&\quad + 1899q^{30} + 1715q^{29} + 1545q^{28} + 1371q^{27} + 1212q^{26} + 1055q^{25} \\
&\quad + 918q^{24} + 783q^{23} + 669q^{22} + 561q^{21} + 470q^{20} + 385q^{19} \\
&\quad + 317q^{18} + 254q^{17} + 205q^{16} + 161q^{15} + 127q^{14} + 97q^{13} \\
&\quad + 76q^{12} + 56q^{11} + 42q^{10} + 30q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}$$

$$
\begin{aligned}
\chi^a_{3,6} &= 1703636 \\
p^a_{3,6}(q) &= 1q^{108} + 1q^{107} + 3q^{106} + 6q^{105} + 12q^{104} + 21q^{103} \\
&\quad + 39q^{102} + 62q^{101} + 102q^{100} + 157q^{99} + 240q^{98} + 350q^{97} \\
&\quad + 509q^{96} + 710q^{95} + 985q^{94} + 1328q^{93} + 1772q^{92} + 2310q^{91} \\
&\quad + 2986q^{90} + 3778q^{89} + 4740q^{88} + 5845q^{87} + 7141q^{86} + 8593q^{85} \\
&\quad + 10255q^{84} + 12067q^{83} + 14087q^{82} + 16243q^{81} + 18581q^{80} + 21017q^{79} \\
&\quad + 23601q^{78} + 26221q^{77} + 28932q^{76} + 31616q^{75} + 34316q^{74} + 36912q^{73} \\
&\quad + 39457q^{72} + 41814q^{71} + 44050q^{70} + 46035q^{69} + 47832q^{68} + 49323q^{67} \\
&\quad + 50587q^{66} + 51502q^{65} + 52165q^{64} + 52472q^{63} + 52516q^{62} + 52211q^{61} \\
&\quad + 51665q^{60} + 50790q^{59} + 49706q^{58} + 48342q^{57} + 46808q^{56} + 45047q^{55} \\
&\quad + 43174q^{54} + 41127q^{53} + 39022q^{52} + 36807q^{51} + 34581q^{50} + 32300q^{49} \\
&\quad + 30060q^{48} + 27808q^{47} + 25635q^{46} + 23494q^{45} + 21455q^{44} + 19478q^{43} \\
&\quad + 17626q^{42} + 15851q^{41} + 14211q^{40} + 12662q^{39} + 11247q^{38} + 9926q^{37} \\
&\quad + 8737q^{36} + 7636q^{35} + 6657q^{34} + 5763q^{33} + 4976q^{32} + 4264q^{31} \\
&\quad + 3647q^{30} + 3092q^{29} + 2618q^{28} + 2197q^{27} + 1840q^{26} + 1526q^{25} \\
&\quad + 1265q^{24} + 1036q^{23} + 849q^{22} + 687q^{21} + 556q^{20} + 443q^{19} \\
&\quad + 355q^{18} + 278q^{17} + 219q^{16} + 169q^{15} + 131q^{14} + 99q^{13} \\
&\quad + 76q^{12} + 56q^{11} + 42q^{10} + 30q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

EXAMPLE C.46. For $N \in [3]$, $n = 6$ and $k = 5$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n)$ are given by

$$
\begin{aligned}
\chi_{1,6}^k &= 672 \\
p_{1,6}^k(q) &= 2q^{30} + 2q^{29} + 4q^{28} + 6q^{27} + 10q^{26} + 13q^{25} \\
&\quad + 19q^{24} + 22q^{23} + 29q^{22} + 33q^{21} + 39q^{20} + 41q^{19} \\
&\quad + 47q^{18} + 46q^{17} + 48q^{16} + 46q^{15} + 45q^{14} + 40q^{13} \\
&\quad + 38q^{12} + 31q^{11} + 28q^{10} + 22q^{9} + 18q^{8} + 13q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0} \\
\chi_{2,6}^k &= 39183 \\
p_{2,6}^k(q) &= 3q^{60} + 4q^{59} + 10q^{58} + 17q^{57} + 33q^{56} + 51q^{55} \\
&\quad + 83q^{54} + 119q^{53} + 176q^{52} + 238q^{51} + 324q^{50} + 415q^{49} \\
&\quad + 534q^{48} + 651q^{47} + 793q^{46} + 929q^{45} + 1083q^{44} + 1219q^{43} \\
&\quad + 1367q^{42} + 1486q^{41} + 1609q^{40} + 1694q^{39} + 1774q^{38} + 1813q^{37} \\
&\quad + 1847q^{36} + 1835q^{35} + 1817q^{34} + 1761q^{33} + 1700q^{32} + 1606q^{31} \\
&\quad + 1515q^{30} + 1398q^{29} + 1290q^{28} + 1165q^{27} + 1050q^{26} + 927q^{25} \\
&\quad + 820q^{24} + 707q^{23} + 612q^{22} + 518q^{21} + 440q^{20} + 363q^{19} \\
&\quad + 302q^{18} + 243q^{17} + 198q^{16} + 156q^{15} + 124q^{14} + 95q^{13} \\
&\quad + 75q^{12} + 55q^{11} + 42q^{10} + 30q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0} \\
\chi_{3,6}^k &= 706364 \\
p_{3,6}^k(q) &= 4q^{90} + 6q^{89} + 16q^{88} + 30q^{87} + 60q^{86} + 101q^{85} \\
&\quad + 175q^{84} + 270q^{83} + 423q^{82} + 618q^{81} + 896q^{80} + 1238q^{79} \\
&\quad + 1699q^{78} + 2238q^{77} + 2923q^{76} + 3703q^{75} + 4641q^{74} + 5670q^{73} \\
&\quad + 6861q^{72} + 8116q^{71} + 9514q^{70} + 10943q^{69} + 12469q^{68} + 13973q^{67} \\
&\quad + 15531q^{66} + 16997q^{65} + 18459q^{64} + 19778q^{63} + 21034q^{62} + 22094q^{61} \\
&\quad + 23055q^{60} + 23777q^{59} + 24374q^{58} + 24722q^{57} + 24927q^{56} + 24886q^{55} \\
&\quad + 24717q^{54} + 24312q^{53} + 23800q^{52} + 23093q^{51} + 22303q^{50} + 21355q^{49} \\
&\quad + 20365q^{48} + 19254q^{47} + 18137q^{46} + 16947q^{45} + 15776q^{44} + 14569q^{43} \\
&\quad + 13414q^{42} + 12246q^{41} + 11149q^{40} + 10069q^{39} + 9069q^{38} + 8101q^{37} \\
&\quad + 7222q^{36} + 6380q^{35} + 5627q^{34} + 4919q^{33} + 4292q^{32} + 3710q^{31} \\
&\quad + 3206q^{30} + 2739q^{29} + 2340q^{28} + 1978q^{27} + 1671q^{26} + 1395q^{25} \\
&\quad + 1166q^{24} + 960q^{23} + 793q^{22} + 645q^{21} + 526q^{20} + 421q^{19} \\
&\quad + 340q^{18} + 267q^{17} + 212q^{16} + 164q^{15} + 128q^{14} + 97q^{13} \\
&\quad + 75q^{12} + 55q^{11} + 42q^{10} + 30q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

EXAMPLE C.47. For $N \in [3]$, $n = 6$ and $k = 4$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,6}^k &= 490 \\
p_{1,6}^k(q) &= 1q^{26} + 2q^{25} + 5q^{24} + 7q^{23} + 12q^{22} + 16q^{21} + 22q^{20} + 26q^{19} \\
&\quad + 33q^{18} + 35q^{17} + 39q^{16} + 39q^{15} + 40q^{14} + 37q^{13} \\
&\quad + 36q^{12} + 30q^{11} + 27q^{10} + 22q^{9} + 18q^{8} + 13q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0} \\
\chi_{2,6}^k &= 20915 \\
p_{2,6}^k(q) &= 1q^{52} + 2q^{51} + 7q^{50} + 15q^{49} \\
&\quad + 32q^{48} + 53q^{47} + 90q^{46} + 135q^{45} + 199q^{44} + 271q^{43} \\
&\quad + 363q^{42} + 456q^{41} + 566q^{40} + 668q^{39} + 776q^{38} + 869q^{37} \\
&\quad + 960q^{36} + 1022q^{35} + 1078q^{34} + 1106q^{33} + 1122q^{32} + 1109q^{31} \\
&\quad + 1088q^{30} + 1040q^{29} + 990q^{28} + 921q^{27} + 851q^{26} + 770q^{25} \\
&\quad + 695q^{24} + 611q^{23} + 537q^{22} + 462q^{21} + 397q^{20} + 333q^{19} \\
&\quad + 280q^{18} + 228q^{17} + 187q^{16} + 149q^{15} + 119q^{14} + 92q^{13} \\
&\quad + 73q^{12} + 54q^{11} + 41q^{10} + 30q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0} \\
\chi_{3,6}^k &= 294966 \\
p_{3,6}^k(q) &= 1q^{78} + 2q^{77} + 7q^{76} + 17q^{75} + 40q^{74} + 78q^{73} \\
&\quad + 148q^{72} + 248q^{71} + 406q^{70} + 619q^{69} + 912q^{68} + 1277q^{67} \\
&\quad + 1747q^{66} + 2289q^{65} + 2935q^{64} + 3646q^{63} + 4439q^{62} + 5262q^{61} \\
&\quad + 6138q^{60} + 6993q^{59} + 7856q^{58} + 8657q^{57} + 9415q^{56} + 10068q^{55} \\
&\quad + 10655q^{54} + 11104q^{53} + 11462q^{52} + 11678q^{51} + 11794q^{50} + 11771q^{49} \\
&\quad + 11664q^{48} + 11426q^{47} + 11121q^{46} + 10714q^{45} + 10257q^{44} + 9726q^{43} \\
&\quad + 9177q^{42} + 8575q^{41} + 7974q^{40} + 7350q^{39} + 6743q^{38} + 6133q^{37} \\
&\quad + 5558q^{36} + 4990q^{35} + 4466q^{34} + 3962q^{33} + 3502q^{32} + 3067q^{31} \\
&\quad + 2681q^{30} + 2319q^{29} + 2002q^{28} + 1711q^{27} + 1459q^{26} + 1231q^{25} \\
&\quad + 1038q^{24} + 863q^{23} + 718q^{22} + 590q^{21} + 484q^{20} + 391q^{19} \\
&\quad + 318q^{18} + 252q^{17} + 201q^{16} + 157q^{15} + 123q^{14} + 94q^{13} \\
&\quad + 73q^{12} + 54q^{11} + 41q^{10} + 30q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}$$

EXAMPLE C.48. For $N \in [4]$, $n = 6$ and $k = 3$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$\begin{aligned}
\chi_{1,6}^k &= 358\\
p_{1,6}^k(q) &= 2q^{22} + 4q^{21} + 8q^{20} + 11q^{19}\\
&\quad + 19q^{18} + 22q^{17} + 28q^{16} + 30q^{15} + 33q^{14} + 32q^{13}\\
&\quad + 33q^{12} + 28q^{11} + 26q^{10} + 21q^{9} + 18q^{8} + 13q^{7}\\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{2,6}^k &= 11205\\
p_{2,6}^k(q) &= 3q^{44} + 8q^{43} + 22q^{42} + 40q^{41} + 76q^{40} + 119q^{39} + 180q^{38} + 243q^{37}\\
&\quad + 324q^{36} + 393q^{35} + 471q^{34} + 531q^{33} + 591q^{32} + 627q^{31}\\
&\quad + 661q^{30} + 666q^{29} + 669q^{28} + 648q^{27} + 625q^{26} + 583q^{25}\\
&\quad + 545q^{24} + 491q^{23} + 444q^{22} + 389q^{21} + 342q^{20} + 291q^{19}\\
&\quad + 250q^{18} + 206q^{17} + 172q^{16} + 138q^{15} + 112q^{14} + 87q^{13}\\
&\quad + 70q^{12} + 52q^{11} + 40q^{10} + 29q^{9} + 22q^{8} + 15q^{7}\\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{3,6}^k &= 123766\\
p_{3,6}^k(q) &= 4q^{66} + 12q^{65} + 36q^{64} + 77q^{63} + 155q^{62} + 268q^{61}\\
&\quad + 446q^{60} + 668q^{59} + 971q^{58} + 1317q^{57} + 1734q^{56} + 2168q^{55}\\
&\quad + 2652q^{54} + 3111q^{53} + 3587q^{52} + 4009q^{51} + 4417q^{50} + 4746q^{49}\\
&\quad + 5045q^{48} + 5246q^{47} + 5408q^{46} + 5473q^{45} + 5496q^{44} + 5428q^{43}\\
&\quad + 5330q^{42} + 5153q^{41} + 4959q^{40} + 4705q^{39} + 4445q^{38} + 4144q^{37}\\
&\quad + 3853q^{36} + 3534q^{35} + 3235q^{34} + 2924q^{33} + 2638q^{32} + 2349q^{31}\\
&\quad + 2091q^{30} + 1835q^{29} + 1611q^{28} + 1394q^{27} + 1207q^{26} + 1030q^{25}\\
&\quad + 881q^{24} + 740q^{23} + 624q^{22} + 517q^{21} + 430q^{20} + 350q^{19}\\
&\quad + 288q^{18} + 230q^{17} + 186q^{16} + 146q^{15} + 116q^{14} + 89q^{13}\\
&\quad + 70q^{12} + 52q^{11} + 40q^{10} + 29q^{9} + 22q^{8} + 15q^{7}\\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}$$

$$
\begin{aligned}
\chi^k_{4,6} &= 783319 \\
p^k_{4,6}(q) &= 5q^{88} + 16q^{87} + 50q^{86} + 114q^{85} \\
&\quad + 244q^{84} + 445q^{83} + 774q^{82} + 1231q^{81} + 1880q^{80} + 2698q^{79} \\
&\quad + 3754q^{78} + 4982q^{77} + 6449q^{76} + 8051q^{75} + 9837q^{74} + 11677q^{73} \\
&\quad + 13625q^{72} + 15523q^{71} + 17440q^{70} + 19226q^{69} + 20950q^{68} + 22472q^{67} \\
&\quad + 23879q^{66} + 25031q^{65} + 26031q^{64} + 26754q^{63} + 27305q^{62} + 27577q^{61} \\
&\quad + 27687q^{60} + 27531q^{59} + 27235q^{58} + 26710q^{57} + 26074q^{56} + 25249q^{55} \\
&\quad + 24354q^{54} + 23311q^{53} + 22235q^{52} + 21057q^{51} + 19878q^{50} + 18637q^{49} \\
&\quad + 17427q^{48} + 16183q^{47} + 14993q^{46} + 13799q^{45} + 12672q^{44} + 11559q^{43} \\
&\quad + 10525q^{42} + 9517q^{41} + 8593q^{40} + 7705q^{39} + 6899q^{38} + 6133q^{37} \\
&\quad + 5448q^{36} + 4801q^{35} + 4229q^{34} + 3696q^{33} + 3229q^{32} + 2797q^{31} \\
&\quad + 2424q^{30} + 2080q^{29} + 1787q^{28} + 1519q^{27} + 1293q^{26} + 1088q^{25} \\
&\quad + 919q^{24} + 764q^{23} + 638q^{22} + 525q^{21} + 434q^{20} + 352q^{19} \\
&\quad + 288q^{18} + 230q^{17} + 186q^{16} + 146q^{15} + 116q^{14} + 89q^{13} \\
&\quad + 70q^{12} + 52q^{11} + 40q^{10} + 29q^{9} + 22q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

EXAMPLE C.49. For $N \in [4]$, $n = 6$ and $k = 2$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}^k_N(\widehat{\mathfrak{gl}}_n)$ are given by

$$
\begin{aligned}
\chi^k_{1,6} &= 262 \\
p^k_{1,6}(q) &= 1q^{20} + 2q^{19} + 7q^{18} + 10q^{17} + 16q^{16} + 20q^{15} + 24q^{14} + 25q^{13} \\
&\quad + 28q^{12} + 25q^{11} + 24q^{10} + 20q^{9} + 17q^{8} + 13q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0} \\
\chi^k_{2,6} &= 6021 \\
p^k_{2,6}(q) &= 1q^{40} + 2q^{39} + 9q^{38} + 20q^{37} \\
&\quad + 45q^{36} + 74q^{35} + 117q^{34} + 161q^{33} + 212q^{32} + 256q^{31} \\
&\quad + 303q^{30} + 336q^{29} + 367q^{28} + 382q^{27} + 391q^{26} + 385q^{25} \\
&\quad + 377q^{24} + 354q^{23} + 332q^{22} + 301q^{21} + 272q^{20} + 238q^{19} \\
&\quad + 209q^{18} + 177q^{17} + 150q^{16} + 123q^{15} + 101q^{14} + 80q^{13} \\
&\quad + 65q^{12} + 49q^{11} + 38q^{10} + 28q^{9} + 21q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0} \\
\chi^k_{3,6} &= 52132 \\
p^k_{3,6}(q) &= 1q^{60} + 2q^{59} + 9q^{58} + 22q^{57} + 55q^{56} + 108q^{55} \\
&\quad + 202q^{54} + 322q^{53} + 487q^{52} + 674q^{51} + 891q^{50} + 1113q^{49} \\
&\quad + 1352q^{48} + 1573q^{47} + 1794q^{46} + 1984q^{45} + 2156q^{44} + 2285q^{43} \\
&\quad + 2392q^{42} + 2448q^{41} + 2481q^{40} + 2469q^{39} + 2434q^{38} + 2361q^{37} \\
&\quad + 2276q^{36} + 2159q^{35} + 2038q^{34} + 1898q^{33} + 1758q^{32} + 1607q^{31} \\
&\quad + 1465q^{30} + 1316q^{29} + 1179q^{28} + 1043q^{27} + 919q^{26} + 799q^{25} \\
&\quad + 695q^{24} + 594q^{23} + 508q^{22} + 428q^{21} + 360q^{20} + 298q^{19} \\
&\quad + 248q^{18} + 201q^{17} + 164q^{16} + 131q^{15} + 105q^{14} + 82q^{13} \\
&\quad + 65q^{12} + 49q^{11} + 38q^{10} + 28q^{9} + 21q^{8} + 15q^{7} \\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

$$
\begin{aligned}
\chi^k_{4,6} &= 270729\\
p^k_{4,6}(q) &= 1q^{80} + 2q^{79} + 9q^{78} + 22q^{77} + 57q^{76} + 118q^{75} + 236q^{74} + 414q^{73}\\
&+ 691q^{72} + 1047q^{71} + 1514q^{70} + 2057q^{69} + 2690q^{68} + 3370q^{67}\\
&+ 4112q^{66} + 4859q^{65} + 5631q^{64} + 6372q^{63} + 7096q^{62} + 7755q^{61}\\
&+ 8370q^{60} + 8887q^{59} + 9340q^{58} + 9684q^{57} + 9950q^{56} + 10104q^{55}\\
&+ 10185q^{54} + 10156q^{53} + 10064q^{52} + 9879q^{51} + 9641q^{50} + 9330q^{49}\\
&+ 8986q^{48} + 8583q^{47} + 8165q^{46} + 7710q^{45} + 7251q^{44} + 6771q^{43}\\
&+ 6302q^{42} + 5822q^{41} + 5363q^{40} + 4906q^{39} + 4473q^{38} + 4051q^{37}\\
&+ 3660q^{36} + 3281q^{35} + 2936q^{34} + 2608q^{33} + 2311q^{32} + 2033q^{31}\\
&+ 1786q^{30} + 1555q^{29} + 1353q^{28} + 1167q^{27} + 1005q^{26} + 858q^{25}\\
&+ 733q^{24} + 618q^{23} + 522q^{22} + 436q^{21} + 364q^{20} + 300q^{19}\\
&+ 248q^{18} + 201q^{17} + 164q^{16} + 131q^{15} + 105q^{14} + 82q^{13}\\
&+ 65q^{12} + 49q^{11} + 38q^{10} + 28q^{9} + 21q^{8} + 15q^{7}\\
&+ 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

EXAMPLE C.50. For $N \in [4]$, $n = 6$ and $k = 1$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N^k\big(\widehat{\mathfrak{gl}}_n\big)$ are given by

$$
\begin{aligned}
\chi_{1,6}^k &= 192\\
p_{1,6}^k(q) &= 2q^{18} + 3q^{17} + 7q^{16} + 10q^{15} + 15q^{14} + 17q^{13}\\
&\quad + 21q^{12} + 20q^{11} + 21q^{10} + 18q^{9} + 16q^{8} + 12q^{7}\\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{2,6}^k &= 3243\\
p_{2,6}^k(q) &= 3q^{36} + 6q^{35} + 16q^{34} + 28q^{33} + 48q^{32} + 69q^{31}\\
&\quad + 98q^{30} + 122q^{29} + 153q^{28} + 174q^{27} + 196q^{26} + 207q^{25}\\
&\quad + 219q^{24} + 216q^{23} + 215q^{22} + 203q^{21} + 193q^{20} + 174q^{19}\\
&\quad + 159q^{18} + 138q^{17} + 122q^{16} + 102q^{15} + 86q^{14} + 69q^{13}\\
&\quad + 58q^{12} + 44q^{11} + 35q^{10} + 26q^{9} + 20q^{8} + 14q^{7}\\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{3,6}^k &= 22024\\
p_{3,6}^k(q) &= 4q^{54} + 9q^{53} + 25q^{52} + 48q^{51} + 87q^{50} + 135q^{49}\\
&\quad + 205q^{48} + 278q^{47} + 371q^{46} + 464q^{45} + 569q^{44} + 663q^{43}\\
&\quad + 766q^{42} + 846q^{41} + 927q^{40} + 983q^{39} + 1034q^{38} + 1056q^{37}\\
&\quad + 1076q^{36} + 1066q^{35} + 1055q^{34} + 1020q^{33} + 984q^{32} + 929q^{31}\\
&\quad + 879q^{30} + 812q^{29} + 752q^{28} + 682q^{27} + 620q^{26} + 551q^{25}\\
&\quad + 492q^{24} + 429q^{23} + 377q^{22} + 323q^{21} + 278q^{20} + 233q^{19}\\
&\quad + 199q^{18} + 163q^{17} + 136q^{16} + 110q^{15} + 90q^{14} + 71q^{13}\\
&\quad + 58q^{12} + 44q^{11} + 35q^{10} + 26q^{9} + 20q^{8} + 14q^{7}\\
&\quad + 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

$$
\begin{aligned}
\chi^k_{4,6} &= 93885 \\
p^k_{4,6}(q) &= 5q^{72} + 12q^{71} + 34q^{70} + 68q^{69} + 128q^{68} + 207q^{67} \\
&+ 324q^{66} + 460q^{65} + 637q^{64} + 830q^{63} + 1060q^{62} + 1296q^{61} \\
&+ 1562q^{60} + 1819q^{59} + 2094q^{58} + 2345q^{57} + 2602q^{56} + 2820q^{55} \\
&+ 3033q^{54} + 3198q^{53} + 3352q^{52} + 3452q^{51} + 3538q^{50} + 3571q^{49} \\
&+ 3592q^{48} + 3561q^{47} + 3521q^{46} + 3437q^{45} + 3348q^{44} + 3221q^{43} \\
&+ 3095q^{42} + 2938q^{41} + 2788q^{40} + 2615q^{39} + 2450q^{38} + 2270q^{37} \\
&+ 2105q^{36} + 1927q^{35} + 1766q^{34} + 1600q^{33} + 1451q^{32} + 1299q^{31} \\
&+ 1166q^{30} + 1032q^{29} + 917q^{28} + 803q^{27} + 705q^{26} + 610q^{25} \\
&+ 532q^{24} + 454q^{23} + 391q^{22} + 331q^{21} + 282q^{20} + 235q^{19} \\
&+ 199q^{18} + 163q^{17} + 136q^{16} + 110q^{15} + 90q^{14} + 71q^{13} \\
&+ 58q^{12} + 44q^{11} + 35q^{10} + 26q^{9} + 20q^{8} + 14q^{7} \\
&+ 11q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

EXAMPLE C.51. For $N \in [4]$ and $n = 6$ the Euler characteristics and Poincaré polynomials of the approximations $\mathrm{Gr}_N(\widehat{\mathfrak{gl}}_n)$ are given by

$$\begin{aligned}
\chi_{1,6} &= 141\\
p_{1,6}(q) &= 1q^{18} + 1q^{17} + 3q^{16} + 5q^{15} + 8q^{14} + 10q^{13}\\
&\quad + 14q^{12} + 14q^{11} + 16q^{10} + 15q^{9} + 14q^{8} + 11q^{7}\\
&\quad + 10q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{2,6} &= 1751\\
p_{2,6}(q) &= 1q^{36} + 1q^{35} + 3q^{34} + 6q^{33} + 11q^{32} + 17q^{31}\\
&\quad + 28q^{30} + 38q^{29} + 53q^{28} + 66q^{27} + 81q^{26} + 92q^{25}\\
&\quad + 105q^{24} + 110q^{23} + 116q^{22} + 116q^{21} + 116q^{20} + 110q^{19}\\
&\quad + 105q^{18} + 95q^{17} + 87q^{16} + 76q^{15} + 66q^{14} + 55q^{13}\\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^{9} + 18q^{8} + 13q^{7}\\
&\quad + 10q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}\\
\chi_{3,6} &= 9331\\
p_{3,6}(q) &= 1q^{54} + 1q^{53} + 3q^{52} + 6q^{51} + 11q^{50} + 18q^{49}\\
&\quad + 31q^{48} + 45q^{47} + 67q^{46} + 92q^{45} + 124q^{44} + 157q^{43}\\
&\quad + 198q^{42} + 235q^{41} + 277q^{40} + 313q^{39} + 350q^{38} + 378q^{37}\\
&\quad + 406q^{36} + 422q^{35} + 438q^{34} + 443q^{33} + 446q^{32} + 439q^{31}\\
&\quad + 432q^{30} + 415q^{29} + 398q^{28} + 374q^{27} + 351q^{26} + 323q^{25}\\
&\quad + 297q^{24} + 267q^{23} + 241q^{22} + 213q^{21} + 188q^{20} + 162q^{19}\\
&\quad + 141q^{18} + 119q^{17} + 101q^{16} + 84q^{15} + 70q^{14} + 57q^{13}\\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^{9} + 18q^{8} + 13q^{7}\\
&\quad + 10q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}$$

$$
\begin{aligned}
\chi_{4,6} &= 32661 \\
p_{4,6}(q) &= 1q^{72} + 1q^{71} + 3q^{70} + 6q^{69} + 11q^{68} + 18q^{67} \\
&\quad + 31q^{66} + 46q^{65} + 70q^{64} + 99q^{63} + 138q^{62} + 183q^{61} \\
&\quad + 241q^{60} + 302q^{59} + 376q^{58} + 452q^{57} + 537q^{56} + 619q^{55} \\
&\quad + 708q^{54} + 788q^{53} + 871q^{52} + 942q^{51} + 1011q^{50} + 1066q^{49} \\
&\quad + 1119q^{48} + 1154q^{47} + 1186q^{46} + 1202q^{45} + 1214q^{44} + 1210q^{43} \\
&\quad + 1203q^{42} + 1181q^{41} + 1157q^{40} + 1120q^{39} + 1081q^{38} + 1032q^{37} \\
&\quad + 984q^{36} + 927q^{35} + 872q^{34} + 812q^{33} + 755q^{32} + 694q^{31} \\
&\quad + 638q^{30} + 579q^{29} + 526q^{28} + 472q^{27} + 423q^{26} + 375q^{25} \\
&\quad + 333q^{24} + 291q^{23} + 255q^{22} + 221q^{21} + 192q^{20} + 164q^{19} \\
&\quad + 141q^{18} + 119q^{17} + 101q^{16} + 84q^{15} + 70q^{14} + 57q^{13} \\
&\quad + 47q^{12} + 37q^{11} + 30q^{10} + 23q^{9} + 18q^{8} + 13q^{7} \\
&\quad + 10q^{6} + 7q^{5} + 5q^{4} + 3q^{3} + 2q^{2} + 1q^{1} + 1q^{0}
\end{aligned}
$$

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